GATE 2021

Quantum Mechanics

1. Three non-interacting bosonic particles of mass m each, are in a one-dimensional infinite potential well of width a. The energy of the third excited state of the system is $x \times \frac{\hbar^2\pi^2}{ma^2}$.
The value of x (in integer) is

Ans: 6

Solution: Ground state is $\frac{3\hbar^2\pi^2}{2ma^2}$ first excited state is

$$2 \times \frac{\hbar^2\pi^2}{2ma^2} + 1 \times \frac{4\hbar^2\pi^2}{2ma^2} = \frac{3\pi^2\hbar^2}{ma^2}$$

Second excited state is

$$1 \times \frac{\hbar^2\pi^2}{2ma^2} + 2 \times \frac{4\hbar^2\pi^2}{2ma^2} = \frac{9\pi^2\hbar^2}{2ma^2}$$

Third excited state is

$$3 \times \frac{4\hbar^2\pi^2}{2ma^2} = \frac{12\pi^2\hbar^2}{2ma^2} = 6\frac{\hbar^2\pi^2}{ma^2}$$

2. Consider a spin $S = \hbar/2$ particle in the state $|\phi\rangle = \frac{1}{3}\begin{bmatrix} 2+i \\ 2 \end{bmatrix}$. The probability that a measurement finds the state with $S_x = +\hbar/2$ is

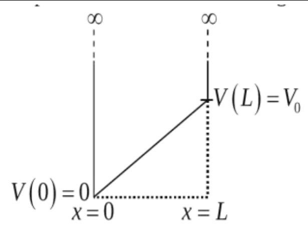
(a) 5/18 (b) 11/18
(c) 15/18 (d) 17/18

Ans: (d)

Solution: $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ the eigen state corresponds to eigen value $S_x = +\hbar/2$ is $|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$P(\hbar/2) = \frac{|\langle u_1 \mid \phi\rangle|^2}{\langle \phi \mid \phi\rangle} = \frac{\left|\frac{1}{3} \cdot \frac{1}{\sqrt{2}}(4+i)\right|^2}{1}$$
$$= \frac{17}{18}$$

3. Consider a particle in a one-dimensional infinite potential well with its walls at $x = 0$ and $x = L$. The system is perturbed as shown in the figure.



The first order correction to the energy eigenvalue is

(a) $\frac{V_0}{4}$ (b) $\frac{V_0}{3}$
(c) $\frac{V_0}{2}$ (d) $\frac{V_0}{5}$

Ans: (c)
Solution:

$$E_n^1 = \int_0^L V_0 \frac{2}{L} \sin^2 \frac{n\pi x}{L} dx = \frac{V_0}{2}$$

4. Consider a state described by $\psi(x,t) = \psi_2(x,t) + \psi_4(x,t)$, where $\psi_2(x,t)$ and $\psi_4(x,t)$ are respectively the second and fourth normalized harmonic oscillator wave functions and ω is the angular frequency of the harmonic oscillator. The wave function $\psi(x, t=0)$ will be orthogonal to $\psi(x,t)$ at time t equal to

(a) $\frac{\pi}{2\omega}$ (b) $\frac{\pi}{\omega}$
(c) $\frac{\pi}{4\omega}$ (d) $\frac{\pi}{6\omega}$

Ans.: (a)
Solution:

$$t = \frac{\hbar}{E_4 - E_2}\cos^{-1}\left(-\frac{|c_4|^2}{|c_2|^2}\right)$$
$$= \frac{\hbar}{2\hbar\omega}\cos^{-1}(-1) = \frac{\pi}{2\omega}$$

5. Consider the potential $U(r)$ defined as

$$U(r) = -U_0\frac{e^{-\alpha r}}{r}$$

where α and U_0 are real constants of appropriate dimensions. According to the first Born approximation, the elastic scattering amplitude calculated with $U(r)$ for a (wavevector) momentum transfer q and $\alpha \to 0$, is proportional to

(Useful integral: $\int_0^\infty \sin(qr) e^{\alpha r} dr = \frac{q}{\alpha^2+q^2}$)

(a) q^{-2} (b) q^{-1}

(c) q (d) q^2

Solution: Scattering amplitude is given by

$$f(\theta) \approx \frac{-2m}{\hbar^2 q}\int_0^\infty rV(r)\sin(qr)dr$$

$$= \frac{-2m}{\hbar^2 q}\int_0^\infty r\left(-U_0\frac{e^{-\alpha r}}{r}\right)\sin(qr)dr$$

$$= \frac{2U_0 m}{\hbar^2 q}\int_0^\infty e^{-\alpha r}\sin\, qrdr$$

$$= \frac{2U_0 m}{2i\hbar^2 q}\left(\int_0^\infty e^{-\alpha r}e^{iqr}dr - \int_0^\infty e^{-\alpha r}e^{-iqr}dr\right)$$

$$= \frac{U_0 m}{i\hbar^2 q}\left(\int_0^\infty e^{-\alpha(r-iq)}dr - \int_0^\infty e^{-r(\alpha+iq)}dr\right)$$

$$f(\theta) = \frac{U_0 m}{i\hbar^2 q}\left(\int_0^\infty e^{-r(\alpha-iq)}dr - \int_0^\infty e^{-r(\alpha+iq)}dr\right)$$

$$= \frac{U_0 m}{i\hbar^2 q}\left(\frac{1}{\alpha-iq}-\frac{1}{\alpha+iq}\right)$$

$$= \frac{U_0 m}{i\hbar^2 q}\left(\frac{\alpha+iq-\alpha+iq}{\alpha^2+q^2}\right) = \frac{2mU_0}{\hbar^2(\alpha^2+q^2)}$$

$$f(\theta) = \frac{2mU_0}{\hbar^2(\alpha^2+q^2)} \Rightarrow \alpha\to 0 \Rightarrow f(\theta)\propto\frac{1}{q^2}$$

6. For a two-nucleon system in spin singlet state, the spin is represented through the Pauli matrices σ_1, σ_2 for particles 1 and 2, respectively. The value of $(\sigma_1\cdot\sigma_2)$ (in integer) is

Ans: -3

Solution: $\vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{|\sigma|^2-|\sigma_1|^2-|\sigma_2|^2}{2}$

For singlet state

$$\sigma = 0, |\sigma_1|^2 = 3, |\sigma_2|^2 = 3 \Rightarrow \vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{0-3-3}{2} = -3$$

7. The normalized radial wave function of the second excited state of hydrogen atom is

$$R(r) = \frac{1}{\sqrt{24}}\left(a^{-3/2}\right)\frac{r}{a}\left(e^{-r/2a}\right)$$

where a is the Bohr radius and r is the distance from the center of the atom. The distance at which the electron is most likely to be found is $y\times a$. The value of y (in integer) is

Ans : 4

Solution: Probability density $\rho(r) = |R(r)|^2 r^2$

For most probable distance

$$\left|\frac{d\rho(r)}{dr}\right| = 0 \Rightarrow \frac{d|r^4\exp-(r/a)|}{dr} = 0 \Rightarrow r = 4a$$

so $y = 4$

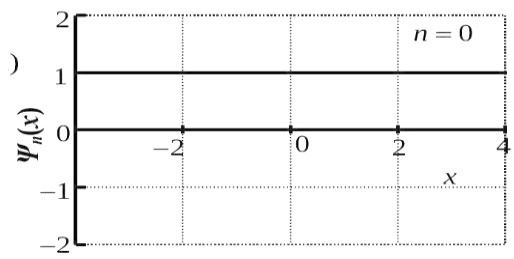
Mathematical Physics

8. If $y_n(x)$ is a solution of the differential equation

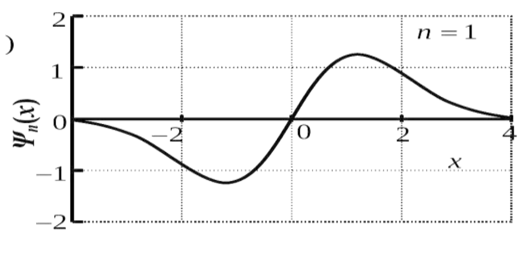
$$y'' - 2xy' + 2ny = 0$$

where n is an integer and the prime (′) denotes differentiation with respect to x, then acceptable plot(s) of $\psi_n(x) = e^{-x^2/2}y_n(x)$, is(are)

(a)



(c)



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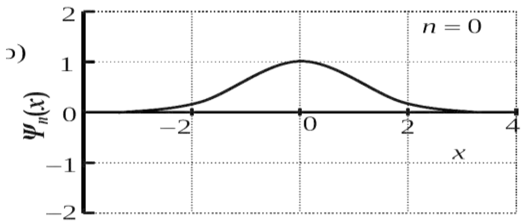
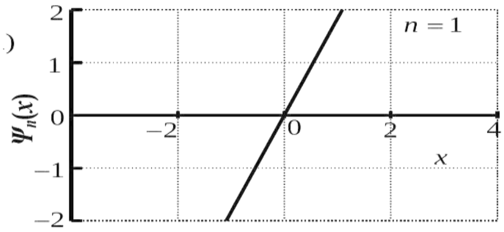


(d)



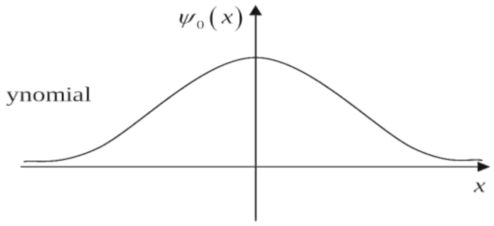
Ans: (b), (c)

Solution: It's a Hermite Differential Equation

$y'' - 2xy' + 2ny = 0, y_n(x)$ is Hermite Polynomial

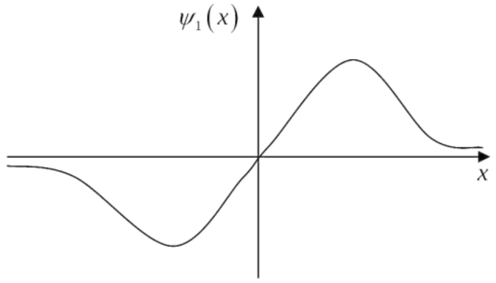
$H_0(x) = 1,\ H_1(x) = 2x$

$\psi_n(x) = e^{-x^2/2} y_n(x)$



for $n = 0: \psi_0(x) = e^{-x^2/2}(1) = e^{-x^2/2}$

For $n = 1: \psi_1(x) = e^{-x^2/2}(2x) = 2xe^{-x^2/2}$



9. A function $f(t)$ is defined only for $t \geq 0$. The Laplace transform of $f(t)$ is

$$L(f; s) = \int_0^\infty e^{-st} f(t) dt$$

whereas the Fourier transform of $f(t)$ is

$$\tilde{f}(\omega) = \int_0^\infty f(t) e^{-i\omega t} dt$$

The correct statement(s) is(are)

(a) The variable s is always real.

(b) The variable s can be complex.

(c) $L(f; s)$ and $\tilde{f}(\omega)$ can never be made connected.

(d) $L(f; s)$ and $\tilde{f}(\omega)$ can be made connected

Ans: (b), (d)

Solution: $h(s) = \int_0^\infty f(t) e^{-st} dt$, where s is a complex number, can be complex or real.

$f(\omega) = \int_0^\infty f(t) e^{-i\omega t} dt(\text{ for } t \geq 0)$

$\Rightarrow$ If $s = j\omega$, then $h(s) = \tilde{f}(\omega)$

So, $L(f, s)$ and $\tilde{f}(\omega)$ can be related.

10. P and Q are two Hermitian matrices and there exists a matrix R, which diagonalizes both of them, such that $RPR^{-1} = S_1$ and $RQR^{-1} = S_2$, where S_1 and S_2 are diagonal matrices. The correct statement(s) is (are)

(a) All the elements of both matrices S_1 and S_2 are real

(b) The matrix PQ can have complex eigenvalues.

(c) The matrix QP can have complex eigenvalues.

(d) The matrices P and Q commute

Ans: (a), (d)

Solution: $P^+ = P,\ Q^+ = Q$

$S_1 = RPR^{-1},\ S_2 = RQR^{-1}$

S_1, S_2 are diagonal matrices of P and Q

As P and Q are Hermitian matrices, their eigenvalues are always Real. Hence their diagonal matrix will only have Real elements.

$\Rightarrow S_1, S_2$ All elements of S_1, S_2 are real.

$\Rightarrow P$ and Q are Simultaneously diagonal, they both commute

$P = R^{-1} S_1 R,\ Q = R^{-1} S_2 R$

$PQ = R^{-1} S_1 R R^{-1} S_2 R = R^{-1} S_1 S_2 R$ as $S_1 S_2 = S_2 S_1$

$QP = R^{-1} S_2 R R^{-1} S_1 R = R^{-1} S_2 S_1 R$ because diagonal matrices commute

$PQ = QP \Rightarrow [PQ] = 0$

P and Q commute.

$\Rightarrow$ As $PQ = R^{-1} S_1 S_2 R,\ QP = R^{-1} S_2 S_1 R$

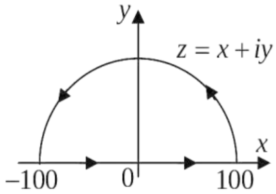
$S_1 S_2, S_2 S_1$ are diagonal matrix of PQ, QP

$\Rightarrow S_1S_2$ have real elements then S_1S_2 and S_2S_1 also have real elements.
Hence PQ and QP have Real eigenvalues NOT complex.

11. A contour integral is defined as

$$I_n = \oint_C \frac{dz}{(z-n)^2+\pi^2}$$

where n is a positive integer and C is the closed contour, as shown in the figure, consisting of the line from -100 to 100 and the semicircle traversed in the counterclockwise sense.



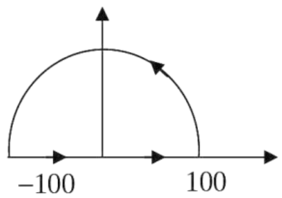
The value of $\sum_{n=1}^{5} I_n$ (in integer) is

Ans: 5

Solution: $I_n = \oint \frac{dz}{(z-n)^2+\pi^2}$

$$I_n = \oint \frac{dz}{(z-n+i\pi)(z-n-i\pi)}$$

Poles $z = n - i\pi, n + i\pi$



$z = n - i\pi$ is NOT allowed outside the contour.
Only one Pole: $z = (n + i\pi)$. It is a simple pole.

$$\lim_{z\to n+i\pi}[z-(n+i\pi)]\frac{1}{[z-(n-i\pi)][z-(n+i\pi)]}$$

$$= \frac{1}{[(n+i\pi)-(n-i\pi)]} = \frac{1}{2i\pi}$$

$$I_n = 2\pi i \times \frac{1}{2\pi i} = 1$$

$$I_1 = I_2 = I_3 = I_4 = I_5 = 1 \Rightarrow \sum_{n=1}^{5} I_n = 1+1+1+1+1 = 5$$

Electromagnetic Theory

12. A matter wave is represented by the wave function

$$\Psi(x,y,z,t) = Ae^{i(4x+3y+5z-10\pi t)}$$

where A is a constant. The unit vector representing the direction of the propagation of this matter wave is

(a) $\frac{4}{5\sqrt{2}}\hat{x} + \frac{3}{5\sqrt{2}}\hat{y} + \frac{1}{\sqrt{2}}\hat{z}$
(b) $\frac{3}{5\sqrt{2}}\hat{x} + \frac{4}{5\sqrt{2}}\hat{y} + \frac{1}{5\sqrt{2}}\hat{z}$
(b) $\frac{1}{5\sqrt{2}}\hat{x} + \frac{3}{5\sqrt{2}}\hat{y} + \frac{1}{\sqrt{2}}\hat{z}$
(d) $\frac{1}{5\sqrt{2}}\hat{x} + \frac{4}{5\sqrt{2}}\hat{y} + \frac{3}{5\sqrt{2}}\hat{z}$

Ans: (a)

Solution: Propagation vector $\vec{k} = 4\hat{x} + 3\hat{y} + 5\hat{z}$
The unit vector representing the direction of the propagation of this matter wave is

$$\hat{k} = \frac{\vec{k}}{|\vec{k}|} = \frac{1}{\sqrt{16+9+25}}(4\hat{x}+3\hat{y}+5\hat{z})$$

$$= \frac{1}{5\sqrt{2}}(4\hat{x}+3\hat{y}+5\hat{z})$$

$$= \frac{4}{5\sqrt{2}}\hat{x} + \frac{3}{5\sqrt{2}}\hat{y} + \frac{1}{\sqrt{2}}\hat{z}$$

13. Consider a tiny current loop driven by a sinusoidal alternating current. If the surface integral of its time-averaged Poynting vector is constant, then the magnitude of the timeaveraged magnetic field intensity, at any arbitrary position, $\vec{r}$, is proportional to

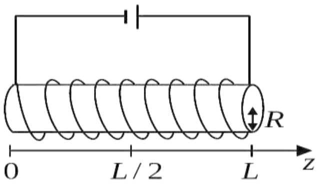
(a) $\frac{1}{r^3}$ (b) $\frac{1}{r^2}$
(c) $\frac{1}{r}$ (d) r

Ans: (c)

Solution: If $|\vec{E}| \propto \frac{1}{r}, |\vec{B}| \propto \frac{1}{r} \Rightarrow |\langle\vec{S}\rangle| \propto \frac{1}{r^2}$.
Then $\oint_S \langle\vec{S}\rangle \cdot d\vec{a} = \text{constant}$

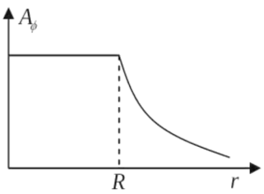
14. Consider a solenoid of length L and radius R, where $R \ll L$. A steady-current flows through the solenoid. The magnetic field is

uniform inside the solenoid and zero outside.



Among the given options, choose the one that best represents the variation in the magnitude of the vector potential, $(0, A_\varphi, 0)$ at $z = L/2$, as a function of the radial distance (r) in cylindrical coordinates.
Useful information: The curl of a vector $\vec{F}$, in cylindrical coordinates is $\vec{\nabla} \times \vec{F}(r, \varphi, z) = \hat{r}\left[\frac{1}{r}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right] + \hat{\varphi}\left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right] + \hat{z}\frac{1}{r}\left[\frac{\partial (rF_\varphi)}{\partial r} - \frac{\partial F_r}{\partial \varphi}\right]$

(a)



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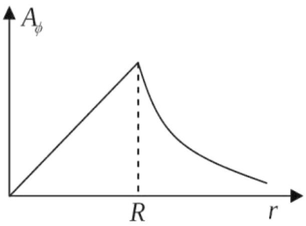

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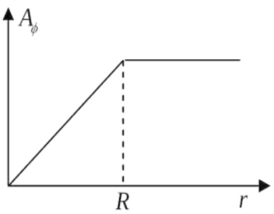

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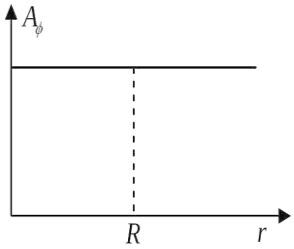


Ans: (c)

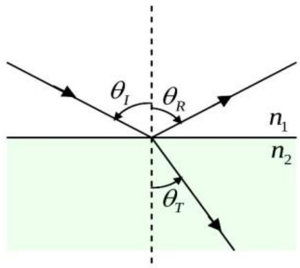
Solution: Since $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \oint_{\text{line}} \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{a}$.

Given that $|\vec{B}|$ = constant (inside) and $|\vec{B}|$ = zero (outside)

For points inside $r < R$: $|\vec{A}| \times 2\pi r =$ Const $\times \pi r^2 \Rightarrow |\vec{A}| \propto r$

For points outside $r > R$: $|\vec{A}| \times 2\pi r =$ Const $\times \pi R^2 \Rightarrow |\vec{A}| \propto \frac{1}{r}$

15. As shown in the figure, an electromagnetic wave with intensity I_I is incident at the interface of two media having refractive indices $n_1 = 1$ and $n_2 = \sqrt{3}$. The wave is reflected with intensity I_R and transmitted with intensity I_T. Permeability of each medium is the same. (Reflection coefficient $R = I_R/I_I$ and Transmission coefficient $T = I_T/I_I$).



Choose the correct statement(s)

(a) $R = 0$ if $\theta_I = 0^0$ and polarization of incident light is parallel to the plane of incidence.

(b) $T = 1$ if $\theta_I = 60°$ and polarization of incident light is parallel to the plane of incidence

(c) $R = 0$ if $\theta_I = 60°$ and polarization of incident light is perpendicular to the plane of incidence

(d) $T=1$ if $\theta_I = 60^\circ$ and polarization of incident light is perpendicular to the plane of incidence

Ans: (b)

Solution: Brewester angle at interface is $\tan\,\theta_B = \frac{n_2}{n_1} = \sqrt{3} \Rightarrow \theta_B = \theta_I = 60^\circ$.

If polarization of incident light is parallel to the plane of incidence then $R=0$ and $T=1$.

If polarization of incident light is perpendicular to the plane of incidence then $R \neq 0$ and $T \neq 1$.

16. A material is placed in a magnetic field intensity H. As a result, bound current density J_b is induced and magnetization of the material is M. The magnetic flux density is B. Choose the correct option(s) valid at the surface of the material

(a) $\nabla \cdot M = 0$ (b) $\nabla \cdot B = 0$

(c) $\nabla \cdot H = 0$ (d) $\nabla \cdot J_b = 0$

Ans.: (b), (d)

17. For a finite system of Fermions where the density of states increases with energy, the chemical potential

(a) Decreases with temperature

(b) Increases with temperature

(c) Does not vary with temperature

(d) Corresponds to the energy where the occupation probability is 0.5

Ans: (a), (d)

Solution: For fermionic system the chemical potential is equal to the fermi energy. The probability function is written as

$$F(E) = \frac{1}{e^{(E-\mu)/k_BT}+1} = \frac{1}{e^{(E-E_F)/k_BT}+1}$$

At finite temperature the chemical potential and hence the fermi energy is written as

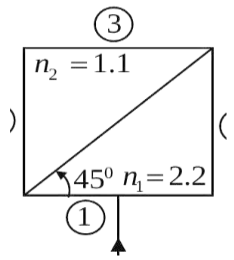
$$\mu = E_F = E_{F_0}\left[1 - \frac{1}{12}\left(\frac{k_BT}{E_{F_0}}\right)^2\right]$$

With increasing temperature, the chemical potential decrease and from probability distribution function it is clear that is also corresponds to the energy where the occupation probability is 0.5 . Thus correct options are (a) and (d).

18. An electromagnetic wave having electric field $E = 8\cos\,(kz - \omega t)\hat{y} V\ \mathrm{cm}^{-1}$ is incident at 90° (normal incidence) on a square slab from vacuum (with refractive index $n_0 = 1.0$) as shown in the figure. The slab is composed of two different materials with refractive indices n_1 and n_2. Assume that the permeability of each medium is the same. After passing through the slab for the first time, the electric field amplitude, in $V\ \mathrm{cm}^{-1}$, of the electromagnetic wave, which emerges from the slab in region 2 , is closest to

(a) $\frac{11}{1.6}$ (b) $\frac{11}{3.2}$

(c) $\frac{11}{25.6}$ (d) $\frac{11}{13.8}$



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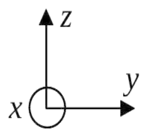


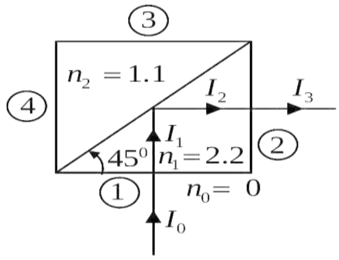
Ans: (a)

Solution: Critical angle at the interface of two slab is

$$\sin\,\theta_C = \frac{n_2}{n_1} = \frac{1.1}{2.2} = \frac{1}{2} \Rightarrow \theta_C = 30^\circ$$

Since incident angle at the interface is $\theta_I = 45^\circ > \theta_C = 30^\circ$, so there is total internal reflection at interface. Thus $I_2 = I_1$.

$$I_0 = \frac{1}{2}c\varepsilon_0 E_0^2 \Rightarrow I_0 = \frac{1}{2}c\varepsilon_0 (8)^2$$
$$= 32 c\varepsilon_0 V/cm$$



At interface 1; $R_1 = \left(\frac{n_0 - n_1}{n_0 + n_1}\right)^2$

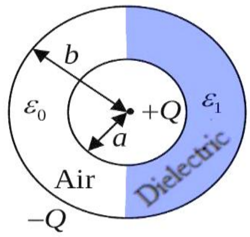
$$\Rightarrow R_1 = \left(\frac{1-2.2}{1+2.2}\right)^2 = \left(\frac{1.2}{3.2}\right)^2 = \frac{9}{64} \Rightarrow T_1$$
$$= 1 - R_1 = 1 - \frac{9}{64} = \frac{55}{64}$$
$$\Rightarrow I_1 = T_1 I_0 = \frac{55}{64} I_0 = I_2$$

At interface 2; $R_2 = \left(\frac{n_1 - n_0}{n_1 + n_0}\right)^2$

$$\Rightarrow R_2 = \left(\frac{2.2-1}{2.2+1}\right)^2 = \frac{9}{64} \Rightarrow T_2 = 1 - R_2$$
$$= 1 - \frac{9}{64} = \frac{55}{64} \Rightarrow I_3 = T_2 I_2$$
$$= \frac{55}{64} I_2 = \frac{55}{64} \times \frac{55}{64} I_0$$
$$I_3 = \frac{1}{2} c\varepsilon_0 E_{03}^2 \Rightarrow \frac{55}{64} \times \frac{55}{64} I_0 = \frac{1}{2} c\varepsilon_0 E_{03}^2$$
$$\Rightarrow \frac{55}{64} \times \frac{55}{64} \times 32 c\varepsilon_0$$
$$= \frac{1}{2} c\varepsilon_0 E_{03}^2$$

$$\Rightarrow E_{03} = \frac{55}{64} \times 8 \Rightarrow E_{03} = \frac{11}{1.6}$$

19. Consider two concentric conducting spherical shells as shown in thee figure. The inner shell has a radius a and carries a charge $+Q$. The outer shell has a radius b and carries a charge $-Q$. The empty space between them is half-filled by a hemispherical shell of a dielectric having permittivity ε_1. The remaining space between the shells is filled with air having the permittivity ε_0.



The electric field at a radial distance r from the center and between the shells ($a < r < b$) is

(a) $\frac{Q}{2\pi(\varepsilon_0+\varepsilon_1)} \frac{\hat{r}}{r^2}$ everywhere

(b) $\frac{Q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2}$ on the air side and $\frac{Q}{4\pi\varepsilon_1} \frac{\hat{r}}{r^2}$ on the dielectric side

(c) $\frac{Q}{2\pi\varepsilon_0} \frac{\hat{r}}{r^2}$ on the air side and $\frac{Q}{2\pi\varepsilon_1} \frac{\hat{r}}{r^2}$ on the dielectric side

(d) $\frac{Q}{4\pi(\varepsilon_0+\varepsilon_1)} \frac{\hat{r}}{r^2}$ everywhere

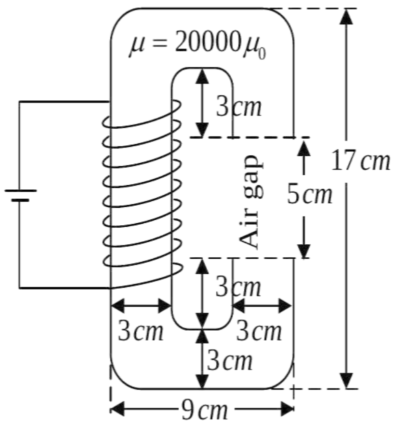
Ans: (a)

Solution: From Gauss Law: $\oint_S \vec{D} \cdot d\vec{a} = Q_{\text{fenc}}$

Let us find the electric field at a radial distance r from the center and between the shells ($a < r < b$). Draw a Gaussian sphere of radius r, then

$$\varepsilon_0 E \times \frac{4\pi r^2}{2} + \varepsilon_1 E \times \frac{4\pi r^2}{2} = Q \Rightarrow \vec{E}$$
$$= \frac{Q}{2\pi(\varepsilon_0 + \varepsilon_1)} \frac{\hat{r}}{r^2}$$

20. Consider a cross-section of an electromagnet having an air-gap of 5 cm as shown in the figure. It consists of a magnetic material ($\mu = 20000\mu_0$) and is driven by a coil having $NI = 10^4$ A, where N is the number of turns and I is the current in Ampere.



Ignoring the fringe fields, the magnitude of the magnetic field $\vec{B}$ (in Tesla, rounded off to two decimal places) in the air-gap between the magnetic poles is

Ans: 0.25

Solution:

$$NI = H_c l_c + H_g l_g \Rightarrow NI = \frac{B_c}{\mu_c} l_c + \frac{B_g}{\mu_g} l_g$$

$$\Rightarrow NI = \left(\frac{1}{\mu_c} l_c + \frac{1}{\mu_g} l_g\right) B_g$$

let $B_c = B_g$.

l_c and l_g are core and gap length. $\mu_c = 20000\mu_0$ and $\mu_g = \mu_0$.

Thus

$$NI = \left(\frac{l_c}{\mu_c} + \frac{l_g}{\mu_g}\right) B_g \Rightarrow 10^4 A$$

$$= \left(\frac{47 \times 10^{-2} m}{2 \times 10^4 \mu_0} + \frac{5 \times 10^{-2}}{\mu_0}\right) B_g$$

$$\Rightarrow B_g = 10^4 \times \mu_0$$

$$= \left(\frac{47 \times 10^{-6} m}{2} + 5 \times 10^{-2}\right)$$

$$\Rightarrow B_g = \frac{10^4 \times \mu_0}{5 \times 10^{-2}} \Rightarrow B_g = \frac{10^4 \times 4\pi \times 10^{-7}}{5 \times 10^{-2}}$$

$$= \frac{0.4 \times 3.14}{5} = 0.25$$

Tesla

Classical Mechanics

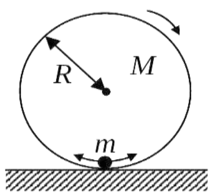
21. Two observers O and O' observe two events P and Q. The observers have a constant relative speed of $0.5c$. In the units, where the speed of light, c, is taken as unity, the observer O obtained the following coordinates:
Event $P: x = 5,\ y = 3,\ z = 5,\ t = 3$
Event $Q: x = 5,\ y = 1,\ z = 3,\ t = 5$
The length of the space-time interval between these two events, as measured by O', is L. The value of $|L|$ (in integer) is

Ans: 2

Solution: $|L| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2} = \sqrt{0 + 2^2 + 2^2 - 1^2 \cdot 2^2} = 2$

22. A hoop of mass M and radius R rolls without slipping along a straight line on a horizontal surface as shown in the figure. A point mass m slides without friction along thee inner surface of the hoop, performing small oscillations about the mean position. The number of degrees of freedom of the system (in integer) is

Ans: 2



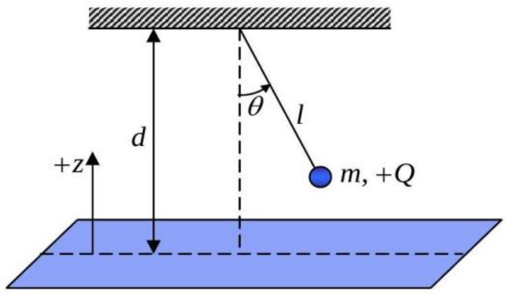
Solution: Two independent motion one is pure rotation and other is pure translation in one dimension. So degrees of freedom is two.

23. If $\vec{a}$ and $\vec{b}$ are constant vectors, $\vec{r}$ and $\vec{p}$ are generalized positions and conjugate momenta, respectively, then for the transformation $Q = \vec{a} \cdot \vec{p}$ and $P = \vec{b} \cdot \vec{r}$ to be canonical, the value of $\vec{a} \cdot \vec{b}$ (in integer) is

Ans: -1

Solution: $[Q,P]=1=[\vec{a}\cdot\vec{p},\vec{b}\cdot\vec{r}]=1\Rightarrow -\vec{a}\cdot\vec{b}=1\Rightarrow\vec{a}\cdot\vec{b}=-1$
answer: $-2/3$)

24. Consider a point charge $+Q$ of mass m suspended by a massless, inextensible string of length l in free space (permittivity ε_0) as shown in the figure. It is placed at a height $d(d>l)$ over an infinitely large, grounded conducting plane. The gravitational potential energy is assumed to be zero at the position of the conducting plane and is positive above the plane.



if θ represents the angular position and p_θ its corresponding canonical momentum, then the correct Hamiltonian of the system is

(a) $\frac{p_\theta^2}{2ml^2}-\frac{Q^2}{16\pi\varepsilon_0(d-l\cos\theta)}-mg(d-l\cos\theta)$

(b) $\frac{p_\theta^2}{2ml^2}-\frac{Q^2}{8\pi\varepsilon_0(d-l\cos\theta)}+mg(d-l\cos\theta)$

(c) $\frac{p_\theta^2}{2ml^2}-\frac{Q^2}{8\pi\varepsilon_0(d-l\cos\theta)}-mg(d-l\cos\theta)$

(d) $\frac{p_\theta^2}{2ml^2}-\frac{Q^2}{16\pi\varepsilon_0(d-l\cos\theta)}+mg(d-l\cos\theta)$

Ans: (d)

Solution: Potential energy due to gravitation $mg(d-l\cos\theta)$

Potential energy due to conducting plate is $\frac{1}{4\pi\varepsilon_0}\cdot\frac{-Q^2}{4(d-l\cos\theta)}$ (we can use a concept of image problem)

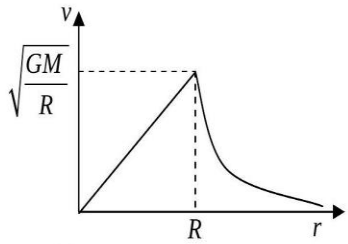
Kinetic energy is $\frac{p_\theta^2}{2ml^2}$.

So Hamiltonian is

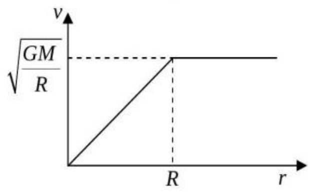
$$H=\frac{p_\theta^2}{2ml^2}-\frac{Q^2}{16\pi\varepsilon_0(d-l\cos\theta)}+mg(d-l\cos\theta)$$

25. Consider a spherical galaxy of total mass M and radius R, having a uniform matter distribution. In this idealized situation, the orbital speed v of a star of mass $m(m\ll M)$ as a function of the distance r from the galactic centre is best described by (G is the universal gravitational constant)

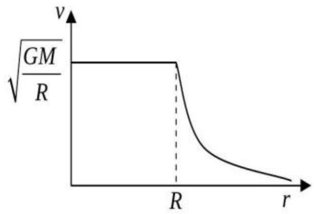
(a)



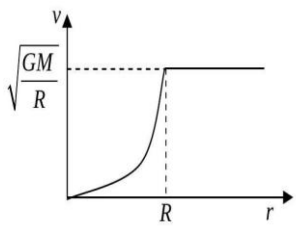
(c)



(b)



(d)



Ans: (a)

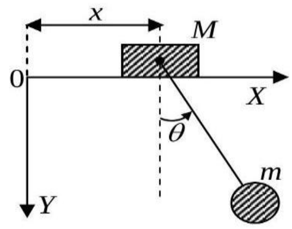
Solution: Gravitational field of galaxy can be given by

$$E(r)=\begin{cases}\frac{GM}{R^3}r, & r<R\\ \frac{GM}{r^2}, & r>R\end{cases}$$

For $r<R$; $\frac{mv^2}{r}=\frac{GMm}{R^3}r\Rightarrow v\propto r$

For $r > R; \frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow v \propto \frac{1}{r}$

26. A uniform block of mass M slides on a smooth horizontal bar. Another mass m is connected to it by an inextensible string of length l of negligible mass, and is constrained to oscillate in the $X - Y$ plane only. Neglect the sizes of the masses. The number of degrees of freedom of the system is two and the generalized coordinates are chosen as x and θ, as shown in the figure.



If p_x and p_θ are the generalized momenta corresponding to x and θ, respectively, then the correct option(s) is (are)
(a) $p_x = (m + M)\dot{x} + ml\cos\,\theta\dot{\theta}$
(b) $p_\theta = ml^2\theta - ml\cos\,\theta\dot{x}$
(c) p_x is conserved
(d) p_θ is conserved

Ans : (a), (c)
Solution:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\,\theta\right) + mgl\cos\,\theta$$

x is cyclic coordinates $\frac{\partial L}{\partial x} = 0 \Rightarrow p_x$ is constant.

$p_x = M\dot{x} + m(\dot{x} + l\dot{\theta}\cos\,\theta)$ or $p_x = (m + M)\dot{x} + ml\cos\,\theta\dot{\theta}$

27. The time derivative of a differentiable function $g(q_i, t)$ is added to a Lagrangian $L(q_i, \dot{q}_1, t)$ such that

$$L' = L(q_i, \dot{q}_i, t) + \frac{d}{dt}g(q_i, t)$$

where $q_i, \dot{q}_i, t$ are the generalized coordinates, generalized velocities and time, respectively. Let p_i be the generalized momentum and H the Hamiltonian associated with $L(q_i, \dot{q}_i, t)$. If p_i' and H' are those associated with L', then the correct option(s) is(are)
(a) Both L and L' satisfy the Euler-Lagrange's equations of motion
(b) $p_i' = p_i + \frac{\partial}{\partial q_i}g(q_i, t)$
(c) If p_i is conserved, then p_i' is necessarily conserved
(d) $H' = H + \frac{d}{dt}g(q_i, t)$

Ans: (a), (b)
Solution: $L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}g(q_i, t) = L(q_i, \dot{q}_i, t) + \frac{\partial g}{\partial q_i}\dot{q}_i + \frac{dg}{dt}$

$$\frac{\partial L'}{\partial q} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}}$$

$$= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial\left(\frac{\partial g}{\partial q}\dot{q}\right)}{\partial q} - \frac{d}{dt}\frac{\partial\left(\frac{\partial g}{\partial q}\dot{q}\right)}{\partial \dot{q}} + \frac{\partial\frac{\partial g}{\partial t}}{\partial q} - \frac{d}{dt}\frac{\partial\frac{\partial g}{\partial t}}{\partial \dot{q}}$$

$$= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 g}{\partial q^2}\dot{q} - \frac{d}{dt}\frac{\partial g}{\partial q} + \frac{\partial^2 g}{\partial q\,\partial t} - \frac{d}{dt}0$$

$$= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 g}{\partial q}\dot{q} - \left(\frac{\partial^2 g}{\partial^2 q}\dot{q} + \frac{\partial^2 g}{\partial t\,\partial q}\right) + \frac{\partial^2 g}{\partial q\,\partial t} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

Which is equivalent to Euler Lagrange's equation.

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}g(q_i, t) = L(q_i, \dot{q}_i, t) + \frac{\partial g}{\partial q_i}\dot{q}_i + \frac{dg}{dt}$$

$$p_i' = \left(\frac{\partial L'}{\partial \dot{q}}\right) \Rightarrow \frac{\partial L}{\partial \dot{q}} + \frac{\partial g}{\partial q_i} = p_i + \frac{\partial g}{\partial q_i}$$

Statistical Mechanics

28. A light source having its intensity peak at the wavelength 289.8 nm which is calibrated as 10,000 K which is the temperature of an equivalent black body radiation. Considering the same calibration, the temperature of light source (in K)

having its intensity peak at the wavelength 579.6 nm (rounded off to the nearest integer) is

Ans: 5000

Solution: Using Wein's law to calculate Temperature of the black body.

$\lambda_{\max}T = \mathrm{b}$, b is known as Wein's constant.

$\lambda_{\max 1}T_1 = \lambda_{\max 2}T_2$

$579.6 \times T = 289.8 \times 10000,\ T$

$$= \frac{289.8 \times 10000}{579.6} = 5000\ \mathrm{K}$$

29. Consider a system of three distinguishable particles, each having spin $S = 1/2$ such that $S_z = \pm 1/2$ with corresponding magnetic moments $\mu_z = \pm\mu$. When the system is placed in an external magnetic field H pointing along the z-axis, the total energy of the system is μH. Let x be the state where the first spin has $S_z = 1/2$. The probability of having the state x and the mean magnetic moment (in the $+z$ direction) of the system in state x are

(a) $\frac{1}{3}, \frac{-1}{3}\mu$ (b) $\frac{1}{3}, \frac{2}{3}\mu$

(c) $\frac{2}{3}, \frac{-2}{3}\mu$ (d) $\frac{2}{3}, \frac{1}{3}\mu$

Ans: (a)

Solution:

State index r	Quantum Numbers m_1, m_2, m_3	Total magnetic moment	Total Energy
1	+++	3μ	$-3\mu H$
2	++-	μ	$-\mu H$
3	+-+	μ	$-\mu H$
4	-++	μ	$-\mu H$
5	+--	$-\mu$	μH
7	-+-	$-\mu$	μH
8	--	-3μ	μH

Accessible states 5,6,7. State 5 is the x state. The probability of having the state x and the mean magnetic moment (in the $+z$ direction) of the system in state x are $\frac{1}{3}, \frac{-1}{3}\mu$.

30. Consider a single one-dimensional harmonic oscillator of angular frequency ω, in equilibrium at temperature $T = (k_B\beta)^{-1}$. The states of the harmonic oscillator are all nondegenerate having energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ with equal probability, where n is the quantum number. The Helmholtz free energy of the oscillator is

(a) $\frac{\hbar\omega}{2} + \beta^{-1}\ln\left[1 - \exp(\beta\hbar\omega)\right]$

(b) $\frac{\hbar\omega}{2} + \beta^{-1}\ln\left[1 - \exp(-\beta\hbar\omega)\right]$

(c) $\frac{\hbar\omega}{2} + \beta^{-1}\ln\left[1 + \exp(-\beta\hbar\omega)\right]$

(d) $\beta^{-1}\ln\left[1 - \exp(-\beta\hbar\omega)\right]$

Ans: (b)

Solution: For 1D Harmonic Oscillator $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$$\frac{5\hbar\omega}{2} - , \frac{3\hbar\omega}{2} - , \frac{\hbar\omega}{2} -$$

$$Z = e^{-\beta\frac{\hbar\omega}{2}} + e^{-\beta\frac{3\hbar\omega}{2}} + e^{-\beta\frac{5\hbar\omega}{2}} + \cdots \ldots \ldots$$

$$\Rightarrow Z = \frac{e^{-\beta\frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}$$

Helmholtz free energy $F = -k_BT\ln Z \Rightarrow$

$F = -k_BT\ln\left(\frac{e^{\frac{-\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}\right) \Rightarrow F =$

$-k_BT\left[\ln e^{\frac{-\beta\hbar\omega}{2}} - \ln\left(1 - e^{-\beta\hbar\omega}\right)\right]$

$$\Rightarrow F = (k_BT)\frac{\hbar\omega}{2(k_BT)} + k_BT\ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= \left[\frac{\hbar\omega}{2} + \ln\left(1 - e^{-\beta\hbar\omega}\right)\right]$$

31. A system of two atoms can be in three quantum states having energies $0, \in$ and $2\in$. The system is in equilibrium at temperature $T = (k_B\beta)^{-1}$. Match the following Statistics with the Partition function.

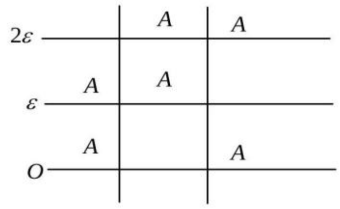
Statistics	Partition function
CD: Classical (distinguishable particles)	Z1: $e^{-\beta\epsilon} + e^{-2\beta\epsilon} + e^{-3\beta\epsilon}$
CI: Classical (indistinguishable particles)	Z2: $1 + e^{-\beta\epsilon} + 2e^{-2\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon}$
FD: Fermi-Dirac	Z3: $1 + 2e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon} + e^{-4\beta\epsilon}$
BE: Bose-Einstein	Z4: $\frac{1}{2} + e^{-\beta\epsilon} + \frac{3}{2}e^{-2\beta\epsilon} + e^{-3\beta\epsilon} + \frac{1}{2}e^{-4\beta\epsilon}$

(a) CD: Z1, CI: Z2, FD: Z3, BE: Z4
(b) CD: Z2, CI: Z3, FD: Z4, BE: Z1
(c) CD: Z3, CI: Z4, FD: Z1, BE: Z2
(d) CD: Z4, CI: Z1, FD: Z2, BE: Z3

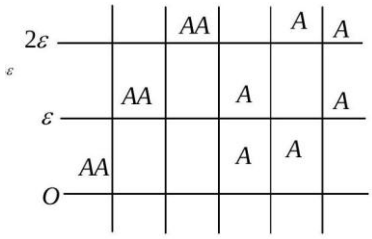
Ans: (c)
Solution: Two particles, Quantum states $0, \varepsilon, 2\varepsilon$
CD: Classical distinguishable (A, B)

$2\varepsilon -$	AB			B	A	B	A
$\varepsilon - AB$		B	A			A	B
OAB			A	B	A	B	

$$Z = 1 + e^{-\beta 2\varepsilon} + e^{-\beta 4\varepsilon} + 2e^{-\beta\varepsilon} + 2e^{-\beta 2\varepsilon} + 2e^{-\beta 3\varepsilon}$$
$$Z \Rightarrow 1 + 2e^{-\beta\varepsilon} + 3e^{-\beta 2\varepsilon} + 2e^{-\beta 3\varepsilon} + e^{-\beta 4\varepsilon}$$
$CD = Z3$
CI : Classical Indistinguishable Particles
$$Z = \frac{(Z3)}{2!} = \frac{1}{2} + e^{-\beta\varepsilon} + \frac{3}{2}e^{-\beta 2\varepsilon} + e^{-\beta 3\varepsilon} + \frac{1}{2}e^{-\beta 4\varepsilon}$$
$CI = Z4$
FD: Fermi Dirac, means they follow Pauli exclusive principle

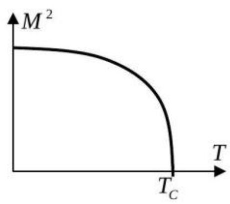


$$Z = e^{-\beta\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta 3\varepsilon}$$
$FD: Z1$
BE : Bose - Einstein (AA)
$$Z = 1 + e^{-\beta 2\varepsilon} + e^{-\beta 4\varepsilon} + e^{-\beta\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta 3\varepsilon}$$
$$Z = 1 + e^{-\beta\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta 3\varepsilon} + e^{-\beta 4\varepsilon}$$
$BE = Z2$
$[CD: Z3, CI: Z4; FD: Z1, BE: Z2]$

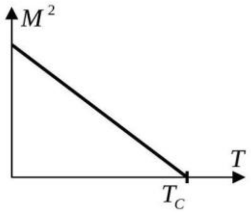


32. The free energy of a ferromagnet is given by $F = F_0 + a_0(T - T_C)M^2 + bM^4$, where F_0, a_0, and b are positive constants, M is the magnetization, T is the temperature, and T_C is the Curie temperature. The relation between M^2 and T is best depicted by

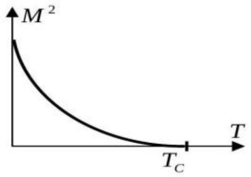
(a)



(b)



(d)



Solution: $\frac{\partial F}{\partial M} = 0 + 2a_0(T - T_C)M + 4bM^3$

Minimising free energy of ferromagnet occurs when $\frac{\partial F}{\partial M} = 0$

$$\frac{\partial F}{\partial M} = 0 + 2a_0(T - T_C)M + 4bM^3 = 0$$

$$(2a_0(T - T_C) + 4bM^2)M = 0$$

Since M can not be zero therefore, first part must be zero

$$2a_0(T - T_C) + 4bM^2 = 0$$

$$M^2 = \frac{-2a_0(T - T_C)}{4b} = \frac{2a_0T_C}{4b} - \frac{2a_0}{4b}T$$

Thus graph (b) correctly represent the variation of M^2 vs temperature.

Atomic Molecular Physics

33. Among the term symbols

$$^4S_1,\ ^2D_{7/2},\ ^3S_1 \text{ and } ^2D_{5/2}$$

Choose the option(s) possible in the LS coupling notation

(a) $^4S_1$ (b) $^2D_{7/2}$

(c) $^3S_1$ (d) $^2D_{5/2}$

Ans: (c), (d)

Solution: (a) $^4S_1 \rightarrow M = 2S + 1 = 4 \rightarrow S = \frac{3}{2}$

$$L = 0 \rightarrow J = \frac{3}{2}$$

So, $^4S_1$ is not possible

(b) $^3D_{7/2} \rightarrow M = 2S + 1 = 3 \rightarrow S = 1$

$$L = 2 \rightarrow J = 1,2,3$$

So, $^3D_{7/2}$ is not possible

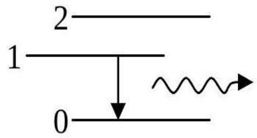
(c) $^3S_1 \rightarrow S = 1, L = 0 \rightarrow J = 1$

so, $^3S_1$ is possible

(d) $^2D_{5/2} \rightarrow S = \frac{1}{2}, L = 2 \rightarrow J = \frac{3}{2}, \frac{5}{2}$

so, $^2D_{5/2}$ is possible

34. To sustain lasing action in a three-level laser as shown in the figure, necessary condition (s) is (are)



(a) Lifetime of the energy level 1 should be greater than that of energy level 2

(b) Population of the particles in level 1 should be greater than that of level 0

(c) Lifetime of the energy level 2 should be greater than that of energy level 0

(d) Population of the particles in level 2 should be greater than that of level 1

Ans: (a), (b)

Solution: (a) Level one should be metastable state for lasing action.

(b) It is necessary for population inversion.

35. The spacing between two consecutive S-branch lines of the rotational Raman spectra of hydrogen gas is 243.2 cm^{-1}. After excitation with a laser of wavelength 514.5 nm, the Stoke's line appeared at 17611.4 cm^{-1} for a particular energy level. The wavenumber (rounded off to the nearest integer), in cm^{-1}, at which Stoke's line will appear for the next higher energy level is

Ans: 17368.2

Solution: The spacing between two consecutive S-branch lines of the rotational Raman spectra

$$4B = 243.2 \text{ cm}^{-1}$$

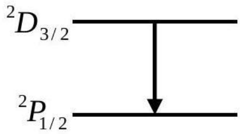
The wavenumber of a stoke's line for a particular energy level $= 17611.4$ cm^{-1}.

The wavenumber of the stoke's line for the next higher energy level

$$= 17611.4 - 4B = 17611.4 - 243.2$$
$$= 17368.2 \text{ cm}^{-1}$$

36. The transition line, as shown in the figure, arises between ${}^2D_{3/2}$ and ${}^2P_{1/2}$ states without any external magnetic field. The number of lines that will appear in the presence of a weak magnetic field (in integer) is

Ans: 6 to 6



Solution: $j_1 = \frac{3}{2}, j_2 = \frac{1}{2}$

$j_1 \neq j_2$

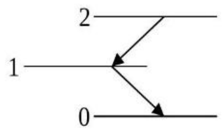
$n_\pi = 2j + 1\; j = \min(j_1, j_2)$

$n_\pi = 2 \cdot \frac{1}{2} + 1 = 2$

$n_{\text{Total}} = 3n_\pi = 3 \times 2 = 6$

$n_\sigma = 6 - 2 = 4$

37. Consider the atomic system as shown in the figure, where the Einstein A coefficients for spontaneous emission for the levels are $A_{2\to1} = 2 \times 10^7 \text{ s}^{-1}$ and $A_{1\to0} = 10^8 \text{ s}^{-1}$. If 10^{14} atoms $/\text{cm}^3$ are excited from level 0 to level 2 and a steady state population in level 2 is achieved, then the steady state population at level 1 will be $x \times 10^{13} \text{ cm}^{-3}$. The value of x (in integer) is



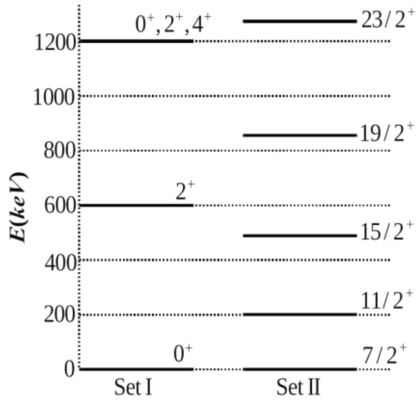
Ans: 2

Solution: $N_2 = 10^{14}$ atoms $/\text{cm}^3$; $\frac{dN_1}{dt} = \frac{N_2}{\tau_2} - \frac{N_1}{\tau_1} = A_{21}N_2 - A_{10}N_1$

In steady state $\frac{dN_1}{dt} = 0$

$$N_1 = \frac{1}{A_{10}} A_{21} N_2 = \frac{10^{14}}{10^8} \times 2 \times 10^7$$
$$= 2 \times 10^{13} \text{ cm}^{-1}$$

38.



For the given sets of energy levels of nuclei X and Y whose mass numbers are odd and even, respectively, choose the best suited interpretation

(a) Set I: Rotational band of X
Set II: Vibrational band of Y

(b) Set I: Rotational band of Y
Set II: Vibrational band of X

(c) Set I: Vibrational band of X
Set II: Rotational band of Y

(d) Set I: Vibrational band of Y
Set II: Rotational band of X

Ans: (d)

Solution: Out of these two is rotational band and other is vibrational band I. For even-even nuclei, the ground state is always $0^+$and first excited state is $2^+$for vibrational and rotational bands. However, in case of vibrational band, next excited states are $0^+, 2^+, 4^+$. So, set I represents vibrational band of even Y nuclei.

39. A linear charged particle accelerator is driven by an alternating voltage source operating at 10MHz. Assume that it is used to accelerate electrons. After a few drift-tubes, the electrons attain a velocity $2.9 \times 10^8 \text{ ms}^{-1}$. The minimum length of each drift-tube, in m, to accelerate the electrons further (rounded off to one decimal place) is

Ans: 14.0 to 15.0

Solution: The length of the drift tube should be

$$L=\frac{\beta\lambda}{2}=\frac{v\lambda}{2c}=\frac{v}{2f}=\frac{2.9\times10^{8}}{2\times10\times10^{6}}=14.5\ \mathrm{m}$$

40. The Coulomb energy component in the binding energy of a nucleus is 18.432MeV. If the radius of the uniform and spherical charge distribution in the nucleus is 3fm, the corresponding atomic number (rounded off to the nearest integer) is
(Given: $\frac{e^2}{4\pi\epsilon_0}=1.44\mathrm{MeVfm}$)

Ans: 8

Solution:

$$E_C=\frac{e^2}{4\pi\varepsilon_0}\frac{3Z^2}{5R}\Rightarrow 18\cdot432=1.44\frac{3Z^2}{5\times3}\Rightarrow Z=\sqrt{\frac{18.432\times5}{1.44}}=8$$

41. Consider an atomic gas with number density $n=10^{20}\ \mathrm{m}^{-3}$, in the ground state at 300 K.
The valence electronic configuration of atoms is f^7. The paramagnetic susceptibility of the gas $\chi=m\times10^{-11}$. The value of m (rounded off to two decimal places) is
(Given : Magnetic permeability of free space $\mu_0=4\pi\times10^{-7}\mathrm{Hm}^{-1}$
Bohr magneton $\mu_B=9.274\times10^{-24}\mathrm{Am}^2$
Boltzmann constant $k_{\mathrm{B}}=1.3807\times10^{-23}\mathrm{JK}^{-1}$)

Ans: 5.48

Solution: The magnetic susceptibility of paramagnetic gas is $\chi=\frac{n\mu_0\mu_B^2}{3k_BT}p^2$, where $p=g\sqrt{J(J+1)}$, is called effective number of Bohr magneton and g is Lande-g factor.
According to Hund's rule

$$\mathrm{S}=7\times\frac{1}{2}=\frac{7}{2}$$

-3	-2	-1	0	+1	+2	+3
↑	↑	↑	↑	↑	↑	↑

$$\mathrm{L}=-3-2-1+0+1+2+3=0$$

$\mathrm{J}=\mathrm{L}+\mathrm{S}=\frac{7}{2}$ ($\because f$ shell is more than half filled)

$$\therefore g=1+\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}=2$$

Thus $p=\sqrt{J(J+1)}=2\sqrt{\frac{7}{2}\left(\frac{7}{2}+1\right)}=3\sqrt{7}$

Given, $\chi=m\times10^{-11}, n=10^{20}\ \mathrm{m}^{-3}, T=300\ \mathrm{K}, \mu_0=4\pi\times10^{-7}\mathrm{Hm}^{-1}$,
$\mu_B=9.274\times10^{-24}\mathrm{Am}^2$ and $k_B=1.3807\times10^{-23}\mathrm{JK}^{-1}$

$$m\times10^{-11}=\frac{10^{20}\times4\pi\times10^{-7}\times(9.274\times10^{-24})^2}{3\times1.3807\times10^{-23}\times300}(3\sqrt{7})^2$$

$$=8.697\times10^{-13}\times63=5.48\times10^{-11}$$

$$m=\frac{5.48\times10^{-11}}{10^{-11}}=5.48$$

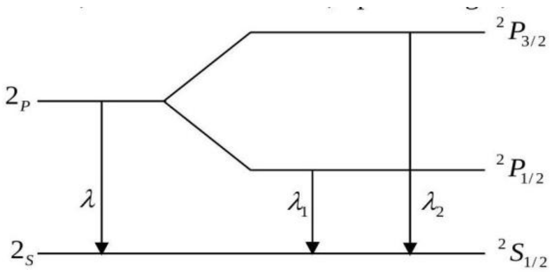
42. The spin $\vec{S}$ and orbital angular momentum $\vec{L}$ of an atom precess about $\vec{J}$, the total angular momentum $\vec{J}$ precesses about an axis fixed by a magnetic field $\vec{B}_1=2B_0\hat{z}$, where B_0 is a constant. Now the magnetic field is changed to $\vec{B}_2=B_0(\hat{x}+\sqrt{2}\hat{y}+\hat{z})$. Given the orbital angular momentum quantum number $l=2$ and spin quantum number $s=1/2, \theta$ is the angle between $\vec{B}_1$ and $\vec{J}$ for the largest possible values of total angular quantum number j and its z-component j_z. The value of θ (in degree, rounded off to the nearest integer) is

Ans: 92 to 93

43. The spin-orbit effect splits the ${}^2P\rightarrow{}^2S$ transition (wavelength, $\lambda=6521\text{Å}$) in Lithium into two lines with separation of $\Delta\lambda=0.14\text{Å}$. The corresponding positive value of energy difference between the above two lines, in eV, is $m\times10^{-5}$. The

value of m (rounded off to the nearest integer) is
(Given: Planck's constant, $h = 4.125 \times 10^{-15}$ eVs, Speed of light, $c = 3 \times 10^8\ \mathrm{ms}^{-1}$)

Ans: 4.08
Solution:



$$E = \frac{hc}{\lambda} \Rightarrow \Delta E = -\frac{hc}{\lambda^2}\Delta\lambda$$

$$= -\frac{1.24 \times 10^{-6}(ev - m)}{(6521 \times 10^{-10}\ \mathrm{m})^2} \times 0.14 \times 10^{-10}\ \mathrm{m}$$

$$\Rightarrow \Delta E = -4.08 \times 10^{-5}\mathrm{eV}$$

Nuclear Physics

44. Assume that $^{13}N(Z = 7)$ undergoes first forbidden β^+decay from its ground state with spin-parity J_i^π, to a final state J_f^π. The possible values for J_i^π and J_f^π, respectively, are

(a) $\frac{1^-}{2}, \frac{5^+}{2}$ (b) $\frac{1^+}{2}, \frac{5^+}{2}$
(c) $\frac{1^-}{2}, \frac{1^-}{2}$ (d) $\frac{1^+}{2}, \frac{1^-}{2}$

Ans: (b)
Solution: $^{13}_{7}N; n(p) = 7 =$
$1s_{1/2}^2, 1p_{3/2}^4, 1p_{1/2}^1 \Rightarrow J_i^\pi = \frac{1^-}{2}$
In first forbidden transition $\Delta\pi =$ yes, $\Delta I = 0, \pm 1, \pm 2$
$\therefore$ Possible values of $\pi_f = +$ ive ; $I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

$$\therefore J_f^\pi = \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2}$$

45. In an experiment, it is seen that an electric-dipole (E1) transition can connect an initial nuclear state of spin-parity $J_i^\pi = 2^+$to a final state J_f^π. All possible values of J_f^π are

(a) $1^+, 2^+$ (b) $1^+, 2^+, 3^+$
(c) $1^-, 2^-$ (d) $1^-, 2^-, 3^-$

Ans: (d)
Solution: In $E1$ transition parity of the nuclei will change, so $\pi_f = -$ ive

$$J_i^\pi = 2^+ \to J_f = 1^-: L = \overbrace{\downarrow}^{1}, 2,3$$
$$J_i^\pi = 2^+ \to J_f = 2^-: L = 0\ \underbrace{E1}_{\substack{\downarrow \\ E}}, 2$$
$$J_i^\pi = 2^+ \to J_f = 3^-: L = \underset{E1}{\overset{E1}{\bigoplus}}, 2,3,4,5$$

So, option (d) is correct.

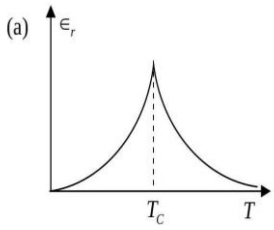
46. The Gell-Mann-Okuba mass formula defines the mass of baryons as $M = M_0 + aY + b\left[I(I+1) - \frac{1}{4}Y^2\right]$, where M_0, a and b are constants, I represents the isospin and Y represents the hypercharge. If the mass of Σ hyperons is same as that of Λ hyperons, then the correct option(s) is(are)

(a) $M \propto I(I+1)$
(b) $M \propto Y$
(c) M does not depend on I
(d) M does not depend on Y

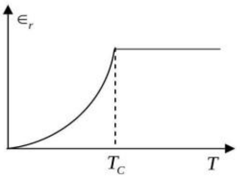
Ans: (b), (c)
Solution: Isospin $I = 1$ for hyperons; $Y = 0$
Isospin $I = 0$ for Λ; $Y = 0$
As isospin for Λ and hyperons are different, but their mass is same. So according to given formula, mass of Baryons is independent of I.
As hypercharge $Y = 0$ for both Λ and hyperons, and their mass is also same. So, one can conclude that mass M is dependent of Y only according to given formula.

Solid State Physics

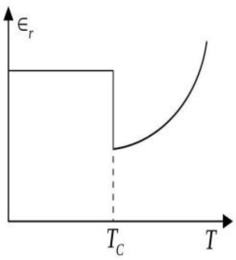
47. Choose the graph that best describes the variation of dielectric constant (ϵ_r) with temperature (T) in ferroelectric material. (T_C is the Curie temperature)
(a)



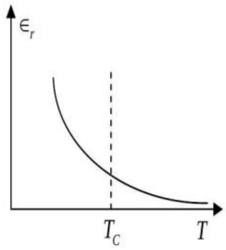
(c)



(b)



(d)



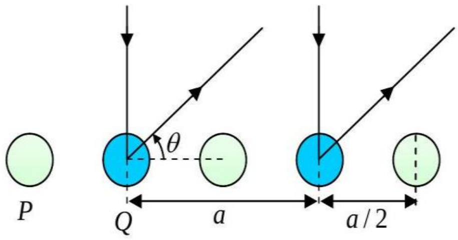
Ans: (a)

Solution: The properties of these materials exist only below a definite phase conversion temperature. Above this temperature, the material will become paraelectric materials. That is, loss in spontaneous polarization. This definite temperature is called Curie temperature (T_C). Most of these materials above T_c will lose the piezoelectric property as well. The variation of dielectric constant (ε_r) by means of temperature (T) in the nonpolar, paraelectric state is shown by Curie-Weiss law as given below

$$\varepsilon_r = \frac{\varepsilon_\infty + A}{T - T_C}$$

Thus the correct option is (a).

48. As shown in the figure, X - ray diffraction pattern is obtained from a diatomic chain of atoms P and Q. The diffraction condition is given by $a\cos\,\theta = n\lambda$, where n is the order of the diffraction peak. Here, a is the lattice constant and λ is the wavelength of the X - rays. Assume that atomic form factors and resolution of the instrument do not depend on θ. Then, the intensity of the diffraction peaks is



(a) Lower for even values of n, when compared to odd values of n
(b) Lower for odd values of n, when compared to even values of n
(d) Zero for odd values of n
(a) Zero for even values of n

Ans: (b)

Solution: Intensity for even values of n is $I = 16(f_P + f_Q)^2$, whereas,
The Intensity for odd values of n is $I = 16(f_P - f_Q)^2$. Where, $f_P \& f_Q$ are the atomic scattering factor of atom P and atom Q. Thus the intensity of diffraction peaks lower for odd values of n, when compared to even values of n. Thus correct option is (b).

49. A two-dimensional square lattice has lattice constant a. k represents the wavevector in reciprocal space. The coordinates (k_x, k_y) of reciprocal space where band gap(s) can occur are

(a) (0,0) (b) $\left(\pm\frac{\pi}{a}, \pm\frac{\pi}{a}\right)$
(c) $\left(\pm\frac{\pi}{a}, \pm\frac{\pi}{1.3a}\right)$ (d) $\left(\pm\frac{\pi}{3a}, \pm\frac{\pi}{a}\right)$

Ans: (b), (c), (d)

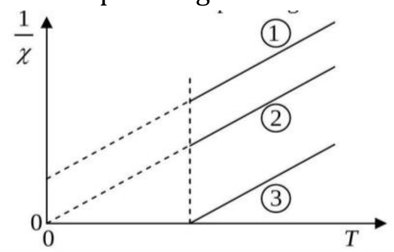
Solution: The band structure of a two-dimensional photonic crystal with a square lattice shows zero band gap at (0,0), while the band gap increases along (k_x, k_y)

directions. Thus correct options are (b), (c) and (d).

50. In a semiconductor, the ratio of the effective mass of hole to electron is 2: 11 and the ratio of average relaxation time for hole to electron is 1: 2. The ratio of the mobility of the hole to electron is
(a) 4: 9 (b) 4: 11
(c) 9: 4 (d) 11: 4

Ans: (d)
Solution: $\mu_h \equiv \frac{q\bar{t}_h}{m_h^*}$ and $\mu_e \equiv \frac{q\overline{t_e}}{m_e^*}$.
Thus $\frac{\mu_h}{\mu_e} \equiv \frac{\overline{t_h}}{m_h^*} \times \frac{m_e^*}{\bar{t}_e} = \frac{m_e^*}{m_h^*} \times \frac{\bar{t}_h}{\bar{t}_e}$.
$\Rightarrow \frac{\mu_h}{\mu_e} = \frac{11}{2} \times \frac{1}{2} = \frac{11}{4}$

51. As shown in the figure, inverse magnetic susceptibility $(1/\chi)$ is plotted as a function of temperature (T) for three different materials in paramagnetic states.



(Curie temperature of ferromagnetic material $= T_C$
Neel temperature of antiferromagnetic material $= T_N$)
Choose the correct statement from the following
(a) Material 1 is paramagnetic, 2 is antiferromagnetic $(T < T_N)$, and 3 is ferromagnetic $(T < T_C)$
(b) Material 1 is antiferromagnetic $(T < T_N)$, 2 is paramagnetic, and 3 is ferromagnetic $(T < T_C)$
(c) Material 1 is ferromagnetic $(T < T_C)$, 2 is antiferromagnetic $(T < T_N)$, and 3 is paramagnetic
(d) Material 1 is ferromagnetic $(T < T_C)$, 2 is paramagnetic, and 3 is antiferromagnetic $(T < T_N)$

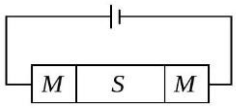
Ans: (b)
Solution: The magnetic susceptibility for various material is written as

1. For paramagnetic: $\chi = \frac{C}{T} \Rightarrow \frac{1}{\chi} = \frac{T}{C}$, thus graph 2 is for paramagnetic materials
2. For Ferromagnetic: $\chi = \frac{C}{T-T_C} \Rightarrow \frac{1}{\chi} = \frac{T-T_C}{C}$, thus graph 3 is for ferromagnetic material.
3. For antiferromagnetic: $\chi = \frac{C}{T+T_N} \Rightarrow \frac{1}{\chi} = \frac{T+T_N}{C}$, thus the graph 1 is for antiferromagnetic material.

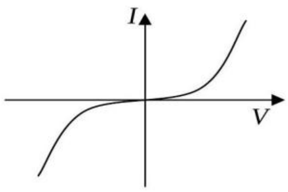
The correct option is (b)

Electronics

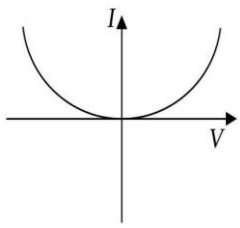
52. As shown in the figure, two metal-semiconductor junctions are formed between an n type semiconductor S and metal M. The work functions of S and M are φ_S and φ_M, respectively with $\varphi_M > \varphi_S$.



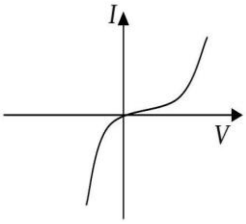
The $I - V$ characteristics (on linear scale) of the junctions is best represented by
(a)



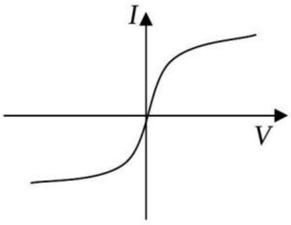
(c)



(b)



(d)



Ans: (a)

53. Choose the correct statement from the following:
(a) Silicon is a direct band gap semiconductor.
(b) Conductivity of metals decreases with increase in temperature.
(c) Conductivity of semiconductor decreases with increase in temperature.
(d) Gallium Arsenide is an indirect band gap semiconductor.

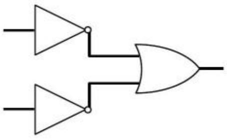
Ans: (b)
Solution: Silicon is indirect and GaAs is a direct bandgap semiconductor. The conductivity of the semiconductor increases with temperature due to increase in carrier concentration. The conductivity of metals decreases with increase in temperature due to increase in number of phonons and hence electron-phonon scattering. Thus correct option is (b)

54. The donor concentration in a sample of n -type silicon is increased by a factor of 100 . Assuming the sample to be non-degenerate, the shift in the Fermi level (in meV) at 300 K (rounded off to the nearest integer) is (Given: $k_B T = 25\text{meV}$ at 300 K)

Ans: 115.15

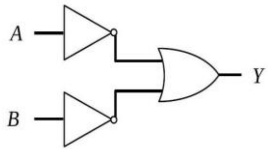
Solution: $E_C - E_F = kT\ln\left(\frac{N_c}{N_d}\right)$ and $E_C - E_F' = kT\ln\left(\frac{N_c}{100N_d}\right) = kT\ln\left(\frac{N_c}{N_d}\right) - kT\ln(100)$
Thus shift is $\Delta E = kT\ln(100) = 25\ln(100)\text{meV} = 115.15\text{meV}$

55.



The above combination of logic gates represents the operation
(a) OR (b) NAND
(c) AND (d) NOR

Ans: (b)
Solution: $Y = \bar{A} + \bar{B} = \overline{AB}$



Answer Key					
1. 6	2. d	3. c	4. a	5.	6. 3
7. 4	8. b,c	9. b,d	10. a,d	11. 5	12. a
13. c	14. c	15. b	16. b,d	17. a,d	18. a
19. a	20. 0.25	21. 2	22. 2	23. -1	24. d
25. a	26. a,c	27. a,b	28. 5000	29. a	30. b
31. c	32.	33. c,d	34. a,b	35. 17368.2	36. 6 to 6
37. 2	38. d	39. 14.0 to 15.0	40. 8	41. 5.48	42. 92 to 93
43. 4.08	44. b	45. d	46. b,c	47. a	48. b
49. b,c,d	50. d	51. b	52. a	53. b	54. 115.15
56. b					