Quantum Mechanics

1. A quantum particle is subjected to the potential

$$V(x)=\begin{cases}\infty, & x\le -\frac{a}{2}\\ 0, & -\frac{a}{2}<x<\frac{a}{2}\\ \infty, & x\ge \frac{a}{2}\end{cases}$$

The ground state wave function of the particle is proportional to

(a) $\sin\left(\frac{\pi x}{2a}\right)$ (b) $\sin\left(\frac{\pi x}{a}\right)$

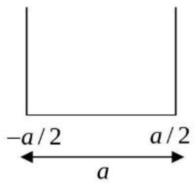
(c) $\cos\left(\frac{\pi x}{2a}\right)$ (d) $\cos\left(\frac{\pi x}{a}\right)$

Ans. : (d)

Solution: Ground state has even parity so

$$|\psi_1\rangle=\sqrt{\frac{2}{a}}\cos\frac{\pi x}{a}$$

Option (d) is correct.



2. Let $\hat{a}$ and $\hat{a}^\dagger$, respectively denote the lowering and raising operators of a one-dimensional simple harmonic oscillator. Let $|n\rangle$ be the energy eigenstate of the simple harmonic oscillator. Given that $|n\rangle$ is also an eigen state of $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$, the corresponding eigenvalue is

(a) $n(n-1)$ (b) $n(n+1)$

(c) $(n+1)^2$ (d) n^2

Ans. : (a)

Solution: $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle=\hat{a}^\dagger\hat{a}^\dagger\hat{a}\sqrt{n}|n-1\rangle$

$=\hat{a}^\dagger\hat{a}^\dagger\sqrt{n}\sqrt{n-1}|n-2\rangle$

$=\hat{a}^\dagger\sqrt{n}\sqrt{n-1}\sqrt{n-1}|n-1\rangle$

$=\sqrt{n}\sqrt{n-1}\sqrt{n-1}\sqrt{n}|n\rangle=n(n-1)|n\rangle$

3. Particle A with angular momentum $j=\frac{3}{2}$ decays into two particles B and C with angular momenta j_1 and j_2, respectively. If $\left|\frac{3}{2},\frac{3}{2}\right\rangle_A=\alpha|1,1\rangle_B\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_C$, the value of α is

Ans: 1

Solution:

4. $\hat{S}_x$ denotes the spin operator defined $\hat{S}_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Which one of the following is correct?

(a) The eigenstates of spin operator $\hat{S}_x$ are $|\uparrow\rangle_x=\begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle_x=\begin{pmatrix}0\\1\end{pmatrix}$

(b) The eigenstates of spin operator $\hat{S}_x$ are $|\uparrow\rangle_x=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and $|\uparrow\rangle_x=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

(c) In the spin state $\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$, upon the measurement of $\hat{S}_x$, the probability for obtaining $|\uparrow\rangle_x$ is $\frac{1}{4}$

(d) In the spin state $\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$, upon the measurement of $\hat{S}_x$, the probability for obtaining $|\uparrow\rangle_x$ is $\frac{2+\sqrt{3}}{4}$

Ans: (d)

Solution: $S_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

Eigen value of $\frac{\hbar}{2},\frac{-\hbar}{2}$ and corresponds eigen state

$\uparrow\rangle_x=|\phi_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, |\downarrow\rangle_x=|\phi_2\rangle$

$=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ | State $|\psi\rangle=\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$

$$\langle\phi_1\mid\psi\rangle=\frac{\left|\frac{1}{\sqrt{2}}(1\ 1)\cdot\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}\right|^2}{\frac{1}{2}(1\sqrt{3})\cdot\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}$$

$$=\left|\frac{1}{2\sqrt{2}}(1+\sqrt{3})\right|^2$$

$$=\frac{(1+\sqrt{3})^2}{8}=\frac{1+3+2\sqrt{3}}{8}=\frac{4+2\sqrt{3}}{8}$$

$$=\frac{2+\sqrt{3}}{4}$$

Option (d) is correct.

5. The radial wave function of a particle in a central potential is give by $R(r) = A\frac{r}{a}\exp\left(-\frac{r}{2a}\right)$, where A is the normalization constant and a is positive constant of suitable dimensions. If γa is the most probable distance of the particle from the force center, the value of γ is

Ans. : 4

Solution: $R(r) = A\frac{r}{a}\exp\left(-\frac{r}{2a}\right)$

Radial probability derivative $p(r) = r^2|R|^2 = \frac{r^4}{a^2}\exp-(r/a)$

For must portable distance $\frac{dp}{dr} = 0$

$$\frac{4r^3}{a^2}e^{-r/a_\omega} + \frac{r^4}{a^2}e^{-r/a}\frac{-1}{a} = 0$$

$$\frac{r^3e^{-r/a_0}}{a^2}\left[4-\frac{r}{a}\right] = 0 \Rightarrow r = 4a = \gamma a \Rightarrow \gamma = 4$$

6. A free particle of mass M is located in a three-dimensional cubic potential well with impenetrable walls. The degeneracy of the fifth excited state of the particle is

Ans. : 6

Solution: Energy eigen value for particle in cubical $= \left(n_x^2+n_y^2+n_z^2\right)E_0$ where $E_0 = \frac{\pi^2\hbar^2}{2ma^2}$.

Ground state $E_{1,1,1} = 3E_0$

First $E_{2,1,1} = E_{1,2,1} = E_{1,1,2} = 6E_0$

Second Excited state $E_{2,2,1} = E_{2,1,2} = E_{1,2,2} = 9E_0$

Third Excited state $E_{3,1,1} = E_{1,3,1} = E_{1,1,3} = 11E_0$

Fourth Excited state $E_{2,2,2} = 12E_0$ non-degenerate

Fifth Excited state $E_{1,2,3} = E_{1,3,2} = E_{2,1,3} = E_{2,3,1} = E_{3,1,2} = E_{3,2,1} = 14E_0$

So fifth excited state has 6 fold degeneracy.

7. An electron in a hydrogen atom is in the state $n=3, l=2, m=-2$. Let $\hat{L}_y$ denote the y-component of the orbital angular momentum operator. If $\left(\Delta\hat{L}_y\right)^2 = \alpha\hbar^2$, the value of α is

Ans: 1

Solution: $\left(\Delta L_y\right) = \sqrt{\langle L_y^2\rangle - \langle L_y\rangle^2}$

$$\langle L_y\rangle = 0$$

$$L_y^2 = \frac{\hbar^2}{2}\left(l(l+1)-m^2\right)l = 2m-2$$

$$= \frac{\hbar^2}{2}(2(3)-4) = \frac{\hbar^2}{2}2\hbar^2$$

$$\left(\Delta L_y\right) = \alpha\hbar^2 \left(\Delta L_y\right)^2 = 1.\hbar^2\ \alpha = 1$$

8. Consider the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'$ where

$$\hat{H}_0 = \begin{pmatrix} E & 0 & 0\\ 0 & E & 0\\ 0 & 0 & E\end{pmatrix}$$

and $\hat{H}'$ is the time independent perturbation given by

$$\hat{H}' = \begin{pmatrix} 0 & k & 0\\ k & 0 & k\\ 0 & k & 0\end{pmatrix}, \text{ where } k > 0.$$

If, the maximum energy eigenvalues of $\hat{H}$ is $3eV$ corresponding to $E = 2eV$, the value of k (rounded off to three decimal places) in eV is check the question

Ans: 1

Solution: $H_0 = \begin{bmatrix} E & 0 & 0\\ 0 & E & 0\\ 0 & 0 & E\end{bmatrix} H' = \begin{pmatrix} 0 & k & 0\\ k & 0 & k\\ 0 & k & 0\end{pmatrix}$

Eigen value of H'

$$\begin{pmatrix} -\lambda & k & 0\\ k & -\lambda & k\\ 0 & x & -\lambda\end{pmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2-k^2) - k(-\lambda k) = 0 \Rightarrow -\lambda(\lambda^2-k^2)+\lambda k^2 = 0$$

$$-\lambda^3+\lambda k^2+\lambda k^2 = 0 \Rightarrow -\lambda^3+2\lambda k^2 = 0$$

$$\Rightarrow \lambda(-\lambda^2+k^2) = 0\ \lambda = 0\ \lambda = k, \lambda = -k$$

$$E_1 = E-k, E_2 = E+0, E_3 = E+k \text{ where } E = 2eV \text{ and } E_3 = 3eV$$

$$E_3 = E+k \Rightarrow k = 3eV - 2eV = 1eV$$

Mathematical Physics

9. Which one of the following is a solution of $\frac{d^2u(x)}{dx^2} = k^2u(x)$, for k real?

(a) e^{-kx} (b) $\sin\ kx$
(c) $\cos\ kx$ (d) $\sinh\ x$

Ans. : (a)
Solution: $m^2 - k^2 = 0 \Rightarrow m = \pm k \Rightarrow u = c_1e^{kx} + c_2e^{-kx}$

10. A real, invertible 3×3 matrix M has eigenvalues $\lambda_i, (i = 1,2,3)$ and the corresponding eigenvectors are $|e_i\rangle, (i = 1,2,3)$ respectively. Which one of the following is correct?

(a) $M|e_i\rangle = \frac{1}{\lambda_i}|e_i\rangle$, for $i = 1,2,3$
(b) $M^{-1}|e_i\rangle = \frac{1}{\lambda_i}|e_i\rangle$, for $i = 1,2,3$
(c) $M^{-1}|e_i\rangle = \lambda_i|e_i\rangle$, for $i = 1,2,3$
(d) The eigenvalues of M and M^{-1} are not related

Ans. : (b)
Solution:

11. For a complex variable z and the contour $c: |z| = 1$ taken in the counter clockwise direction, $\frac{1}{2\pi i}\oint_C \left(z - \frac{2}{z} + \frac{3}{z^2}\right) dz =$

Ans. : -2
Solution: $b_1 = -2$

$$\frac{1}{2\pi i} \times 2\pi i \times -2 = -2$$

12. The product of eigenvalues of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is

(a) -1 (b) 1
(c) 0 (d) 2

Ans. : (a)
Solution: $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |A| = -1$

13. Let $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Let $S = \{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. Let $\mathbb{R}^3$ denote the three dimensional real vector space. Which one of the following is correct?

(a) S is an orthonormal set
(b) S is a linearly dependent set
(c) S is a basis for $\mathbb{R}^3$
(d) $\sum_{i=1}^{3}|e_i\rangle\langle e_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ans. : (c)
Solution: $\langle e_1 \mid e_2\rangle \neq 0$ (i) is false

$$c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$c_1 + c_2 + c_3 = 0$$
$$c_3 = 0, c_2 = 0, c_1 = 0$$
$$c_2 + c_3 = 0$$
$$c_3 = 0$$

$|e_2\rangle, \langle e_5\rangle$ is limits indeed so $\langle 2 \mid 1\rangle$ is correct.
Option (c) is correct.

14. Which one of the following matrices does NOT represent a proper rotation in a plane?

(a) $\begin{pmatrix} -\sin\ \theta & \cos\ \theta \\ -\cos\ \theta & -\sin\ \theta \end{pmatrix}$
(b) $\begin{pmatrix} \cos\ \theta & \sin\ \theta \\ -\sin\ \theta & \cos\ \theta \end{pmatrix}$
(c) $\begin{pmatrix} \sin\ \theta & \cos\ \theta \\ -\cos\ \theta & \sin\ \theta \end{pmatrix}$
(d) $\begin{pmatrix} -\sin\ \theta & \cos\ \theta \\ -\cos\ \theta & \sin\ \theta \end{pmatrix}$

Ans: (d)
Solution: Rotational matrix is orthogonal matrix

(1) $\sin^2\ \theta - (-\cos^2\ \theta) = 1$
(2) $\sin^2\ \theta - (\cos^2\ \theta) = 1$
(3) $-\sin^2\ \theta - (-\cos^2\ \theta) = \cos^2\ \theta - \sin^2\ \theta = \cos^2\ \theta$
(4) $\cos^2\ \theta - (-\sin^2\ \theta) = 1$

15. If $x = \sum_{k=1}^{\infty} a_k \sin\ kx$, for $-\pi \leq x \leq \pi$, the value of a_2 is

Ans: -1
Solution: $x\sin\ kx = a_k\sin^2\ kx$

$$\int_{-\pi}^{\pi} x\sin\ kxdx = a_k \int_{-\pi}^{\pi} \sin^2\ kxdx$$
$$= 2a_k \int_0^{\pi} \sin^2\ kxdx = \frac{2}{k} a_k \int_0^{k\pi}$$
$$\sin^2\ \theta d\theta = \frac{2}{k} a_k \frac{k\pi}{2} = a_k \pi$$
$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin\ 2xdx = \frac{1}{\pi}(-\pi) = -1$$

16. Let $f_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & \frac{1}{2n} < x \end{cases}$

The value of $\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\sin\ xdx$ is

Ans: 0

Solution: $\lim_{n\to\infty}\int_{-1/m}^{1/m} x\sin\ xdx =$ $-n\left[\cos\frac{1}{m} - \cos\frac{-1}{m}\right] = 0$

Electromagnetic Theory

17. A medium ($\varepsilon_r > 1, \mu_r = 1, \sigma > 0$) is semi-transparent to an electromagnetic wave when

(a) Conduction current >> Displacement current

(b) Conduction current << Displacement current

(c) Conduction current = Displacement current

(d) Both Conduction current and Displacement current are zero

Ans: (b)

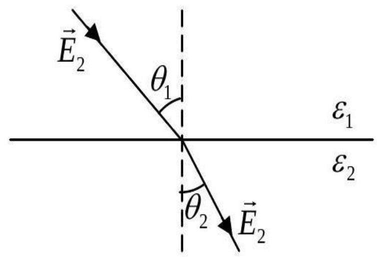
Solution: Conduction current $J_c = \sigma E = \sigma E_0 \cos\ \omega t$

Displacement current $J_d = \varepsilon\frac{\partial E}{\partial t} \Rightarrow |J_d| = \omega\varepsilon E_0 \sin\ \omega t$

For semi-transparent medium i.e for poor conductor $\sigma << \varepsilon$.

Let $\omega t = \frac{\pi}{4} \Rightarrow \frac{J_c}{J_d} = \frac{\sigma E_0}{\omega\varepsilon E_0} = \frac{\sigma}{\omega\varepsilon} << 1 \Rightarrow J_c << J_d$

18. Which one of the following relations determines the manner in which the electric field lines are refracted across the interface between two dielectric media having dielectric constants ε_1 and ε_2 (see figure)?



(a) $\varepsilon_1 \sin\ \theta_1 = \varepsilon_2 \sin\ \theta_2$

(b) $\varepsilon_1 \cos\ \theta_1 = \varepsilon_2 \cos\ \theta_2$

(c) $\varepsilon_1 \tan\ \theta_1 = \varepsilon_2 \tan\ \theta_2$

(d) $\varepsilon_1 \cot\ \theta_1 = \varepsilon_2 \cot\ \theta_2$

Solution: $\frac{\tan\ \theta_1}{\tan\ \theta_2} = \frac{E_1^{\parallel}/E_1^{\perp}}{E_2^{\parallel}/E_2^{\perp}} = \frac{E_2^{\perp}}{E_1^{\perp}}\ \because E_2^{\parallel} = E_1^{\parallel}$

$\because D_1^{\perp} = D_2^{\perp} \Rightarrow \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \Rightarrow \frac{E_2^{\perp}}{E_1^{\perp}} = \frac{\varepsilon_1}{\varepsilon_2}$

$$\Rightarrow \frac{\tan\ \theta_1}{\tan\ \theta_2} = \frac{E_2^{\perp}}{E_1^{\perp}} = \frac{\varepsilon_1}{\varepsilon_2} \Rightarrow \frac{\cot\ \theta_2}{\cot\ \theta_1} = \frac{\varepsilon_1}{\varepsilon_2}$$
$$\Rightarrow \varepsilon_1 \cot\ \theta_1 = \varepsilon_2 \cot\ \theta_2$$

19. If $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively, then $\vec{E}\cdot\vec{B}$ is

(a) odd under parity and even under time reversal

(b) even under parity and odd under time reversal

(c) odd under parity and odd under time reversal

(d) even under parity and even under time reversal

Ans. : (c)

Solution: The electric field $E = -\partial V/\partial r$ changes sign under P-operation as r changes its sign and does not under T-operation as r does not

The magnetic field $B = i \times r$ does not changes its sign under P-operation as $i \rightarrow$

$-i$ and $r \to -r$ under P-operation but under T-operation changes its sign because $i \to -i$ but $r \to r$.
Thus $\vec{E} \cdot \vec{B}$ changes its sign under P-operation as well as under T-operation.
Correct option is (c)

20. Far from the Earth, the Earth's magnetic field can be approximated as due to a bar magnet of magnetic pole strength 4×10^{14}Am. Assume this magnetic field is generated by a current carrying loop encircling the magnetic equator. The current required to do so is about 4×10^n A, where n is an integer. The value of n is
(Earth's circumference: 4×10^7 m)

Ans: 7
Solution: $M = 4 \times 10^{14}$Am and $2\pi R = 4 \times 10^7$ m

$$M \times L = I \times \pi R^2 \Rightarrow M \times 2R = I \times \pi R^2 \Rightarrow I = \frac{M \times 2}{\pi R} = \frac{4 \times 10^{14} \times 2}{2 \times 10^7} = 4 \times 10^7 A \Rightarrow n = 7$$

21. A conducting sphere of radius 1 m is placed in air. The maximum number of electrons that can be put on the sphere to avoid electrical breakdown is about 7×10^n, where n is an integer. The value of n is
Assume:
Breakdown electric field strength in air is $|\vec{E}| = 3 \times 10^6$ V/m
Permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m
Electron charge $e = 1.60 \times 10^{-19}$C

Ans: 14
Solution: $E = \frac{1}{4\pi\varepsilon_0}\frac{Ne}{r^2} < 3 \times 10^6 V/m \Rightarrow$

$$9 \times 10^9 \times N \times \frac{1.6\times10^{-19}}{(1\text{ m})^2} < 3 \times 10^6$$

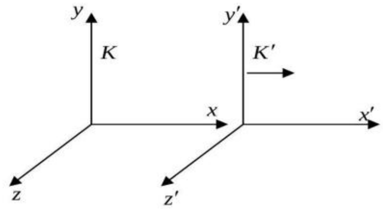
$$\Rightarrow N < \frac{10^6}{4.8} \approx 2 \times 10^{15} = 20 \times 10^{14}$$

$\Rightarrow n \approx 14$

22. A uniform magnetic field $\vec{B} = B_0\hat{y}$ exists in an internal frame K. A perfect conducting sphere moves with a constant velocity $\vec{v} = v_0\hat{x}$ with respect to this inertial frame. The rest frame of the sphere is K' (see figure). The electric and magnetic fields in K and K' are related as

$$\left.\begin{aligned} \vec{E}'_{\parallel} &= \vec{E}_{\parallel} \quad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\ \vec{B}'_{\parallel} &= \vec{B}_{\parallel} \quad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2} \times \vec{E}\right) \end{aligned}\right\}, \gamma = \frac{1}{\sqrt{1-(v/c)^2}}$$

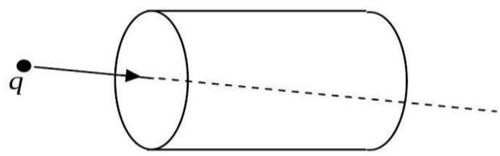
The induced surface charge density on the sphere (to the lowest order in v/c) in the frame K' is



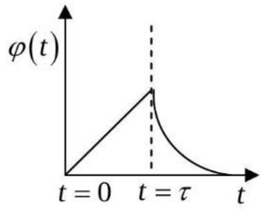
(a) maximum along z'
(b) maximum along y'
(c) maximum along x'
(d) uniform over the sphere

Ans: (a)
Solution:

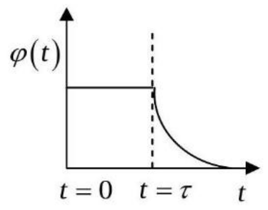
23. A charge q moving with uniform speed enters a cylindrical region in free space at $t = 0$ and exits the region at $t = \tau$ (see figure). Which one of the following options best describes the time dependence of the total electric flux $\varphi(t)$, through the entire surface of the cylinder?



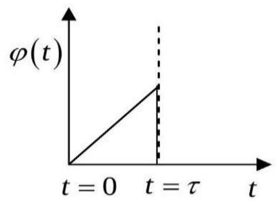
(a)



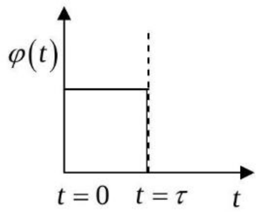
(c)



(b)



(d)



Ans: (d)

Solution: Flux through the closed surface = $\frac{Q_{enc}}{\varepsilon_0}$ = constant when charge is inside otherwise zero.

24. A plane electromagnetic wave of wavelength λ is incident on a circular loop of conducting wire. The loop radius is $a(a < \lambda)$. The angle (in degrees), made by the Poynting vector with the normal to the plane of the loop to generate a maximum induced electrical signal, is

Ans: -270 or -90 or 90 or 270

Classical Mechanics

25. A particle is moving in a central force field given by $\vec{F} = -\frac{k}{r^3}\hat{r}$, where $\hat{r}$ is the unit vector pointing away from the center of the field. The potential energy of the particle is given by

(a) $\frac{k}{r^2}$ (b) $\frac{k}{2r^2}$

(c) $-\frac{k}{r^2}$ (d) $-\frac{k}{2r^2}$

Ans: (d)

Solution: $-\frac{\partial u}{\partial r} = -\frac{k}{r^3} \Rightarrow u = \int \frac{k}{r^3} dr = \frac{kr^{-3+1}}{-3+1} + c$

$$\Rightarrow u = \frac{-k}{2r^2} + c$$

26. A small disc is suspended by a fiber such that it is free to rotate about the fiber axis (see figure). For small angular deflections, the Hamiltonian for the disc is given by

$$H = \frac{p_\theta^2}{2I} + \frac{1}{2}\alpha\theta^2$$

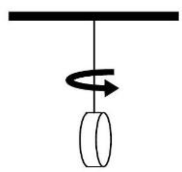
where I is the moment of inertia and α is the restoring torque per unit deflection. The disc is subjected to angular deflections (θ) due to thermal collisions from the surrounding gas at temperature T and p_θ is the momentum conjugate to θ. The average and the rootmean-square angular deflection, θ_{avg} and θ_{rms}, respectively are

(a) $\theta_{\text{avg}} = 0$ and $\theta_{\text{rms}} = \left(\frac{k_BT}{\alpha}\right)^{3/2}$

(b) $\theta_{\text{avg}} = 0$ and $\theta_{\text{rms}} = \left(\frac{k_BT}{\alpha}\right)^{1/2}$

(c) $\theta_{\text{avg}} \neq 0$ and $\theta_{rms} = \left(\frac{k_BT}{\alpha}\right)^{1/2}$

(d) $\theta_{\text{avg}} \neq 0$ and $\theta_{rms} = \left(\frac{k_BT}{\alpha}\right)^{3/2}$



Ans: (b)

Solution:

27. Let p be the momentum conjugate to the generalized coordinate q. If the transformation

$$Q = \sqrt{2}q^m \cos\, p$$
$$P = \sqrt{2}q^m \sin\, p$$

is canonical, then $m =$

Ans: 0.5

Solution: $Q = \sqrt{2}q^m \cos\ p,\ P = \sqrt{2}q^m \sin\ p$

$$\frac{\partial Q}{\partial q} \cdot \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \cdot \frac{\partial P}{\partial q} = 1$$

$$(\sqrt{2}mq^{m-1}\cos\ p)(\sqrt{2}q^m \cos\ p) - (\sqrt{2}q^m(-\sin\ p) \cdot \sqrt{2}mq^{m-1}\sin\ p) = 1$$

$$2mq^{2m-1}(\cos^2\ p + \sin^2\ p) = 1$$

$$2mq^{2m-1} = 1 \Rightarrow 2m - 1 = 0 \text{ or } m = \frac{1}{2} = 0.5$$

28. If a particle is moving along a sinusoidal curve, the number of degree of freedom of the particle is

Ans: 1

Solution: equation of constrain is $y = A\sin\ x$ and $z = 0$

$$DOF = 3.N - k\ N = 1, k = 2$$

$3 - 1 - 2 = 1$ So one degree of freedom

29. Consider the Lagrangian $L = a\left(\frac{dx}{dt}\right)^2 + b\left(\frac{dy}{dt}\right)^2 + cxy$, where a, b and c are constants. If p_x and p_y are the momenta conjugate to the coordinates x and y respectively, then the Hamiltonian is

(a) $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} - cxy$ (b) $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} - cxy$

(c) $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} + cxy$ (d) $\frac{p_x^2}{a} + \frac{p_y^2}{b} + cxy$

Ans. : (a)

Solution: $L = a\dot{x}^2 + b\dot{y}^2 + cxy$

$$\frac{\partial L}{\partial \dot{x}} = p_x = 2a\dot{x} \Rightarrow \dot{x} = \frac{p_x}{2a} \text{ and } \frac{\partial L}{\partial \dot{y}} = p_y = 2a\dot{y} \Rightarrow \dot{y} = \frac{p_y}{2a}$$

$$H = p_x\dot{x} + p_y\dot{y} - L \Rightarrow H = 2a\dot{x}^2 + 2b\dot{y}^2 - (a\dot{x}^2 + b\dot{y}^2 + cxy)$$

$$\Rightarrow H = a\dot{x}^2 + b\dot{y}^2 - cxy = \frac{p_x^2}{4a} + \frac{p_y^2}{4b} - cxy$$

30. The potential energy of a particle of mass m is given by
$U(x) = a\sin\ (k^2x - \pi/2),\ a > 0,\ k^2 > 0$
The angular frequency of small oscillations of the particle about $x = 0$ is

(a) $k^2\sqrt{\frac{2a}{m}}$ (b) $k^2\sqrt{\frac{a}{m}}$

(c) $k^2\sqrt{\frac{a}{2m}}$ (d) $2k^2\sqrt{\frac{a}{m}}$

Ans: (b)

Solution: $U(x) = a\sin\ (k^2x - \pi/2),\ a > 0,\ k^2 > 0$

$$\Rightarrow U(x) = -a\cos^2\ k^2x = -a\left[1 - \frac{k^4x^2}{\lfloor 2} + \cdots\right]$$

$$\Rightarrow F = -\frac{\partial U}{\partial x} = -ak^4x \Rightarrow \omega^2 = \frac{ak^4}{m} \Rightarrow \omega = k^2\sqrt{\frac{a}{m}}$$

31. Let u^μ denote the 4 -velocity of a relativistic particle whose square $u^\mu u_\mu = 1$. If $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor then the value of $\varepsilon_{\mu\nu\rho\sigma}u^\mu u^\nu u^\rho u^\sigma$ is

Ans: 0

Solution:

Statistical Mechanics

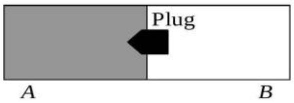
32. Choose the correct statement related to the Fermi energy (E_F) and the chemical potential (μ) of a metal
(a) $\mu = E_F$ only at $0K$
(b) $\mu = E_F$ at finite temperature
(c) $\mu < E_F$ at $0K$
(d) $\mu > E_F$ at finite temperature

Ans.: (a)

Solution: In metal the Fermi energy (E_F) is also known as chemical potential (μ) at absolute zero.

33. As shown in the figure, an ideal gas is confined to chamber A of an insulated

container, with vacuum in chamber B. When the plug in the wall separating the chambers A and B is removed, the gas fills both the chambers. Which one of the following statements is true?



(a) The temperature of the gas remains unchanged
(b) Internal energy of the gas decreases
(c) Temperature of the gas decreases as it expands to fill the space in chamber B
(d) Internal energy of the gas increases as its atoms have more space to move around

Ans: (a)
Solution: Free expansion case, temperature remains unchanged.

34. The internal energy U of a system is given by $U(S,V)=\lambda V^{-2/3}S^2$, where λ is a constant of appropriate dimensions; V and S denote the volume and entropy, respectively. Which one of the following gives the correct equation of state of the system?
(a) $\frac{PV^{1/3}}{T^2}=\text{constant}$
(b) $\frac{PV}{T^{1/3}}=\text{constant}$
(c) $\frac{P}{V^{1/3}T}=\text{constant}$
(d) $\frac{PV^{2/3}}{T}=\text{constant}$

Ans: (a)
Solution: $dU=TdS-PdV$

$$\left(\frac{\partial U}{\partial S}\right)_V=T,\left(\frac{\partial U}{\partial V}\right)_S=-P$$

$$2\lambda V^{-2/3}S=T \text{ and } -\frac{2}{3}\lambda V^{-5/3}S^2=-P$$

$$\Rightarrow \frac{PV}{TS}=\text{ constant } \Rightarrow \frac{PV}{T(TV^{2/3})}=\text{ constant}$$

$$\Rightarrow \frac{PV^{1/3}}{T^2}=\text{ constant}$$

35. For a gas of non-interacting particles, the probability that a particle has a speed v in the internal v to $v+dv$ is given by

$$f(v)dv=4\pi v^2dv\left(\frac{m}{2\pi k_BT}\right)^{3/2}e^{-mv^2/2k_BT}$$

If E is the energy of a particle, then the maximum in the corresponding energy distribution in units of E/k_BT occurs at (rounded off to one decimal place).

Ans: 0.5

Solution: $E_P=\frac{1}{2}k_BT\Rightarrow\frac{\frac{1}{2}k_BT}{k_BT}=0.5$

36. The Planck's energy density distribution is given by $u(\omega)=\frac{\hbar\omega^3}{\pi^2c^3(e^{\hbar\omega/k_BT}-1)}$. At long wavelengths, the energy density of photons in thermal equilibrium with a cavity at temperature T varies as T^α, where α is

Ans: 1

Solution: $u(\omega)=\frac{\hbar\omega^3}{\pi^2c^3\left(1+\frac{\hbar\omega}{k_BT}+\cdots-1\right)}$

$$\frac{kc}{\omega}=1$$

$$\omega=\frac{2\pi c}{\lambda}$$

$u(\omega)\propto T^1$

λ long higher term neglected.

Atomic Molecular Physics

37. The total angular momentum j of the ground state of the $^{17}_{8}O$ nucleus is
(a) $\frac{1}{2}$ (b) 1
(c) $\frac{3}{2}$ (d) $\frac{5}{2}$

Ans: (d)
Solution: For $^{17}_{8}O: z=8$ and $N=9$
For $N=$
$9:(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^1$
The angular momentum is $I=\frac{5}{2}$
Thus the correct option is (d).

38. Consider a diatomic molecule formed by identical atoms. If E_V and E_C represent the energy of the vibrational nuclear motion

and electronic motion respectively, then in terms of the electronic mass m and nuclear mass M, $\frac{E_V}{E_C}$ is proportional to

(a) $\left(\frac{m}{M}\right)^{1/2}$ (b) $\frac{m}{M}$

(c) $\left(\frac{m}{M}\right)^{3/2}$ (d) $\left(\frac{m}{M}\right)^{2}$

Ans: (a)

Solution:

39. A hydrogenic atom is subjected to a strong magnetic field. In the absence of spin-orbit coupling, the number of doubly degenerate states created out of the d-level is

Ans.: 3

Solution: Number of Zeeman levels in strong field can be found from

$$E = (m_L + 2m_s)\mu_B B$$

For d-level: $L = 2$ and $S = 1/2$

$M_L = +2$

$M_S = +1/2$

$M_L + 2m_S = +3$

$\therefore E = +3\mu_B B, +2\mu_B B, +1\mu_B B,$

$+2$	$+1$	$+1$	0	0	-1	-1	-2	-2
$-1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$
$+1$	$+2$	0	$+1$	-1	0	-2	-1	-3
$+1\mu_B B,$	$0,$	$0,$	$-1\mu_B B,$	$-1\mu_B B,$	$-2\mu_B B,$			$-3\mu_B B$

The number of doubly degenerate states are $+\mu_B B,\ 0, -\mu_B B$ only three. Thus correct answer is 3.

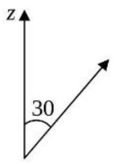
40. A hydrogen atom is in an orbital angular momentum state $|l, m = l\rangle$. If $\vec{L}$ lies on a cone which makes a half angle 30° with respect to the z-axis, the value of l is

Ans: 3

Solution: $\cos\ 30 = \frac{m}{\sqrt{l(l+1)}}\ m = l$

$$\frac{\sqrt{3}}{2} = \frac{l}{\sqrt{l(l+1)}} = \sqrt{3}(\sqrt{l(l+1)}) = 2l$$

$$3(l^2 + l) = 4l^2 \Rightarrow 3l = l^2 \Rightarrow l = 3$$



41. Consider a gas of hydrogen atoms in the atmosphere of the Sun where the temperature is 5800 K. If a sample from this atmosphere contains 6.023×10^{23} of hydrogen atoms in the ground state, the number of hydrogen atoms in the first excited state is approximately 8×10^n, where n is an integer. The value of n is (Boltzmann constant: 8.617×10^{-5}eV/K)

Ans: 14

Solution: $\frac{N_1}{N_0} = e^{-\triangle E/kT}$

$$\Delta E = \left(\frac{13 \cdot 6}{12} - \frac{13 \cdot 6}{2^2}\right) \text{ev} = (13 \cdot 6 - 3 \cdot 4)$$

$$= 10 \cdot 2\text{Ev}$$

$$\therefore \frac{\Delta E}{kT} = \frac{10.2\text{eV}}{8.617 \times 10^{-5}\text{ev/k} \times 5800\text{k}} = 20.41$$

$$\text{Thus } \frac{N_1}{N_0} = e^{-20\cdot 41} \Rightarrow N_1$$

$$= 6 \cdot 023 \times 10^{23} \times 1 \cdot 37 \times 10^{-9}$$

$$\therefore n = 14$$

Nuclear Physics

42. A particle X is produced in the process $\pi^+ + p \to K^+ + X$ via the strong interaction. If the quark content of the K^+ is $u\bar{s}$, the quark content of X is

(a) $c\bar{s}$ (b) und

(c) uus (d) $u\bar{d}$

Ans: (c)

Solution: Lets first identify the particle X

$\pi^+ + p \to K^+ + X$

q: $+1 + 1 + 1 + 1$

spin: 0 1/2 0 1/2

B: 0 + 1 0 + 1

I: 0 1/2 1 1/2

I_3: $+1 + \frac{1}{2}\ + \frac{1}{2} + 1$

S: 0 0 +1 −1

Thus the particle X is Σ^+. The quark content of Σ^+ is uus. Thus the correct option (c)

43. A particle Y undergoes strong decay $Y \to \pi^- + \pi^-$. The isospin of Y is

Ans: 2

Solution: $Y \to \pi^- + \pi^-$

$I:\quad 2 \quad 1 \quad 1$

In strong interaction, isospin is conserved, thus the isospin of Y is 2 .

44. According to the Fermi gas model of nucleus, the nucleons move in a spherical volume of radius $R\left(= R_0 A^{\frac{1}{3}}\right.$, where A is the mass number and R_0 is an empirical constant with the dimensions of length). The Fermi energy of the nucleus E_F is proportional to

(a) R_0^2 (b) $\frac{1}{R_0}$

(c) $\frac{1}{R_0^2}$ (d) $\frac{1}{R_0^3}$

Ans.: (c)

Solution: Fermi energy $E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$

$$V = \frac{4\pi}{3}R^3 = \frac{4\pi}{3}\left(R_0 A^{1/3}\right)^3 = \frac{4\pi}{3}R_0^3 A$$

$$\therefore E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{\frac{4\pi}{3}R_0^3 A}\right)^{2/3}$$

$$= \frac{\hbar^2}{2m}\left(\frac{9\pi N}{4A}\cdot\frac{1}{R_0^3}\right)^{2/3} \Rightarrow E_F \propto \frac{1}{R_0^2}$$

Thus correct option is (c)

45. In the center of mass frame, two protons each having energy 7000GeV, collide to produce protons and anti-protons. The maximum number of anti-protons produced is (Assume the proton mass to be $1\text{GeV}/c^2$)

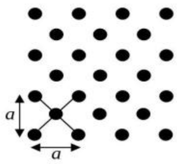
Ans: 6999

Solution: Assuming that protons and anti-protons are produced at rest with mass $1\text{GeV}/c^2$ $p + p \to p + p + n$-number of protons $+n$-number of anti protons

$E: 7000 + 7000 = 1 + 1 + 6999\ + 6999$

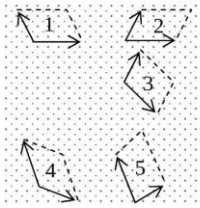
Solid State Physics

46. The number of distinct ways the primitive unit cell can be constructed for the two dimensional lattice as shown in the figure is



Ans.: 5

Solution:



47. Consider a one-dimensional non-magnetic crystal with one atom per unit cell. Assume that the valence electrons (i) do not interact with each other and (ii) interact weakly with the ions. If n is the number of valence electrons per unit cell, then at 0 K,

(a) the crystal is metallic for any value of n

(b) the crystal is non-metallic for any value of n

(c) the crystal is metallic for even values of n

(d) the crystal is metallic for odd values of n

Ans: (d)

Solution: The conduction band is partially filled for odd value of n and hence behaves as a metal.

The band is totally filled for even value of n and known as non-metallic.

Thus correct option is (d)

48. Consider a two dimensional crystal with 3 atoms in the basis. The number of allowed optical branches (n) and acoustic branches (m) due to the lattice vibrations are

(a) $(n, m) = (2,4)$ (b) $(n, m) = (3,3)$

(c) $(n, m) = (4,2)$ (d) $(n, m) = (1,5)$

Ans: (c)

Solution: For p-atoms per basis
Total degree of freedom $= 2p$
Number of acoustical branches $(n) = 2$
Number of optical branches $(n) = 2p - 2$
For $p = 3$
$m = 2$ and $n = 2 \times 3 - 2 = 4$
$\therefore (n, m) = (4,2)$
Thus correct option is (c)

49. Consider a simple cubic monoatomic Bravais lattice which has a basis with vectors $\vec{r}_1 = 0, \vec{r}_2 = \frac{a}{4}(\hat{x} + \hat{y} + \hat{z}), a$ is the lattice parameter. The Bragg reflection is observed due to the change in the wave vector between the incident and the scattered beam as given by $\vec{K} = n_1\vec{G}_1 + n_2\vec{G}_2 + n_3\vec{G}_3$, where $\vec{G}_1, \vec{G}_2$ and $\vec{G}_3$ are primitive reciprocal lattice vectors. For $n_1 = 3, n_2 = 3$ and $n_3 = 2$, the geometrical structure factor is

Ans: 2

Solution: Geometric structure factor $S = \sum_{N=1}^{2} e^{2\pi i(n_1 x_N + n_2 y_N + n_3 z_n)} = 1 + e^{2\pi\left(\frac{3}{4}+\frac{3}{4}+\frac{2}{4}\right)}$

$$= 1 + e^{2\pi i\left(\frac{8}{4}\right)} = 1 + e^{4\pi i} = 1 + 1 = 2$$
$$\therefore S = 2$$

Electronics

50. Which one of the following is a universal logic gate?
(a) AND (b) NOT
(c) OR (d) NAND

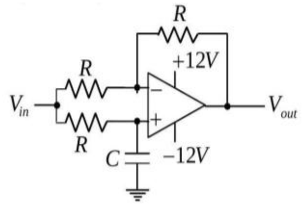
Ans: (d)
Solution:

51. Which one of the following is the correct binary equivalent of the hexadecimal $F6C$?
(a) 011011111100
(b) 111101101100
(c) 110001101111
(d) 011011000111

Ans: (b)

Solution: $F \to (15)_{10} \to (1111)_2, 6 \to (6)_{10} \to (0110)_2$ and $C \to (12)_{10} \to (1100)_2$
Thus $F6C \to (111101101100)_2$

52. The input voltage (V_{in}) to the circuit shown in the figure is $2\cos(100t)V$. The output voltage (V_{out}) is $2\cos\left(100t - \frac{\pi}{2}\right)V$. If $R = 1k\Omega$, the value of C (in μF)is



(a) 0.1 (b) 1
(c) 10 (d) 100

Ans: (c)

Solution: $V_{\text{out}} = -\frac{R}{R}V_{\text{in}} + \left(1 + \frac{R}{R}\right)\left(\frac{X_C}{R+X_C}\right)V_{\text{in}} \Rightarrow V_{\text{out}} = -V_{\text{in}} + 2\left(\frac{1/j\omega C}{R+1/j\omega C}\right)V_{\text{in}}$

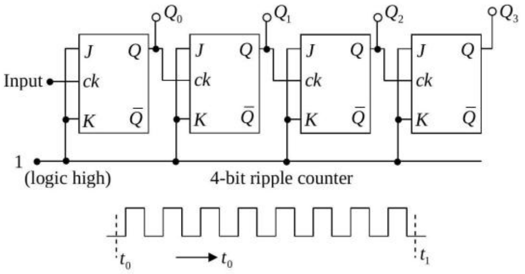
$$\Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} = -1 + 2\left(\frac{1}{j\omega CR + 1}\right) = \left(\frac{1 - j\omega CR}{1 + j\omega CR}\right)$$
$$\Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} = \left(\frac{\sqrt{1 + (\omega CR)^2}}{\sqrt{1 + (\omega CR)^2}}\right)\frac{e^{-j\theta}}{e^{j\theta}}$$
$$= e^{-j2\theta} \text{ where } \theta = \tan^{-1}(\omega RC)$$

Thus $\phi = -2\theta = -2\tan^{-1}(\omega RC) \Rightarrow -\frac{\pi}{2} = -2\tan^{-1}(100 \times 1 \times 10^3 \times C)$

$$\Rightarrow 10^5 \times C = \tan^{-1}\left(\frac{\pi}{4}\right) \Rightarrow C = \frac{1}{10^5}F$$
$$= 10\mu F$$

53. Consider a 4-bit counter constructed out of four flip-flops. It is formed by connecting the J and K inputs to logic high and feeding the Q output to the clock input of the following flip-flop (see the figure). The input signal to the counter is a series of square pulses and the change of state is triggered by the falling edge. At time $t = t_0$ the outputs are in logic low state ($Q_0 =$

$Q_1 = Q_2 = Q_3 = 0$). Then at $t = t_1$, the logic state of the outputs is



Input signal

(a) $Q_0 = 1, Q_1 = 0, Q_2 = 0$ and $Q_3 = 0$

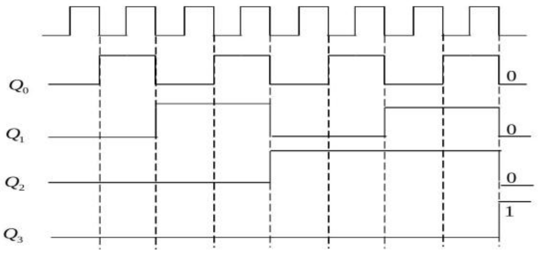
(b) $Q_0 = 0, Q_1 = 0, Q_2 = 0$ and $Q_3 = 1$

(c) $Q_0 = 1, Q_1 = 0, Q_2 = 1$ and $Q_3 = 0$

(d) $Q_0 = 0, Q_1 = 1, Q_2 = 1$ and $Q_3 = 1$

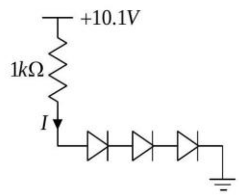
Ans: (b)

Solution:



54. Consider the circuit given in the figure. Let the forward voltage drop across each diode be 0.7 V. The current I (in mA) through the resistor is

Ans: 8



Solution: Current $I = \frac{10.1\text{ V} - 3\times 0.7\text{ V}}{1\text{k}\Omega} = 8\text{ mA}$

55. A sinusoidal voltage of the form $V(t) = V_0 \cos(\omega t)$ is applied across a parallel plate capacitor placed in vacuum. Ignoring the edge effects, the induced emf within the region between the capacitor plates can be expressed as a power series in ω. The lowest nonvanishing exponent in ω is

Ans: 2

Solution: Induced e.m.f $\varepsilon = -\frac{d\phi}{dt} = -\frac{AdB}{dt}$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

Consider an amperian loop of radius $r (r < R)$, then $I_{\text{enc}} = 0$ and since

$$E(t) = \frac{V(t)}{d} = \frac{V_0 \cos\ \omega t}{d}$$

Thus $|\vec{B}| \times 2\pi r = \mu_0 \varepsilon_0 \times \left(-\frac{V_0 \omega \sin\ \omega t}{d}\right) \times \pi r^2 \Rightarrow |\vec{B}| \propto \omega \sin\ \omega t$

$$\Rightarrow \varepsilon \propto \frac{dB}{dt} \propto \omega^2 \cos\ \omega t$$

$$\propto \omega^2 \left(1 - \frac{\omega^2 t^2}{2} + \cdots\right)$$

The lowest non-vanishing exponent in ω is $n = 2$.

Answer Key					
1. d	2. a	3. a	4. d	5. 4	6. 6
7. 1	8. 1	9. a	10. b	11. -2	12. a
13. c	14. d	15. -1	16 0.	17. b	18.
19. c	20. 7	21. 14	22. a	23. d	24.
25. d	26. b	27. 0.5	28. 1	29. a	30. b
31. 0	32. a	33. a	34. a	35. 0.5	36. 1
37. d	38. a	39. 3	40. 3	41. 14	42. c
43. 2	44. c	45. 6999	46. 5	47. d	48. c
49. 2	50. d	51. b	52. c	53. b	54. 8
55.2					

Q.24: -270 or -90 or 90 or 270